

Recall def of dot product.

A nice and useful interpretation of the dot product

Theorem 1. Given any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)$, where θ is the angle between the two vectors.

Sketch of Proof in \mathbb{R}^2 (draw picture): Let $r = \|\vec{v}\|$, $s = \|\vec{w}\|$. Then $\vec{v} = (r \cos(\alpha), r \sin(\alpha))$, and $\vec{w} = (s \cos(\beta), s \sin(\beta))$. So $\vec{v} \cdot \vec{w} = r \cos(\alpha) s \cos(\beta) + r \sin(\alpha) s \sin(\beta)$, which by a trig identity, equals $rs \cos(\alpha - \beta)$. QED.

Example 1. Use the above theorem to find the angle between $\vec{v} = (1, 2, 3)$ and $\vec{w} = (2, 0, -1)$.

Recall def of orthogonal: $\vec{v} \perp \vec{w}$, $\vec{v} \perp V$.

Example 2. Q: Let $\vec{v} = (1, 2) \in \mathbb{R}^2$. We shine light from above perpendicular to the x -axis. Find the shadow of \vec{v} as a vector on the x -axis. *Ans:* The shadow is $\vec{s} = (1, 0)$.

Q: Let $\vec{v} = (1, 2) \in \mathbb{R}^2$. We shine light perpendicular to the line L given by the eqn $y = x/2$. Let's denote the shadow of \vec{v} onto L by \vec{s} . Draw a picture, showing the vectors \vec{v} , \vec{s} , $\vec{v} - \vec{s}$. What is the angle between \vec{s} and $\vec{v} - \vec{s}$? *Ans:* 90 degrees. Why? Because the light is perp to the line.

We use the above observation to give a precise def of "shadow." Instead of "shadow" we say *projection*.

Definition 1. For $\vec{v} \in \mathbb{R}^n$ and W a subspace of \mathbb{R}^n , the **projection** of \vec{v} onto W is defined as: $\text{proj}(\vec{v}, W) =$ a vector $\vec{p} \in W$ such that $\vec{v} - \vec{p}$ is orthogonal to W .

Note. The notation "proj" is not in our book.

Example 3. Let $\vec{v} = (1, 2, 3)$, $W = xy$ -plane. Find $\text{proj}(\vec{v}, W)$. *Ans:* $(1, 2, 0)$.

Definition 2. For $\vec{v}, \vec{w} \in \mathbb{R}^n$, the **projection** of \vec{v} onto \vec{w} is defined as: $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{v}, L)$, where L is the line that contains \vec{w} .

Example 4. Let $\vec{v} = (1, 2)$, $\vec{w} = (0, 5)$. Find $\text{proj}(\vec{v}, \vec{w})$. *Ans:* $L = y$ -axis, so $\text{proj}(\vec{v}, \vec{w}) = (0, 2)$.

Q: In general, given arbitrary vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, is $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{v}, 2\vec{w})$? *Ans:* Yes; \vec{w} and $2\vec{w}$ give the same line L .

Q: In general, given arbitrary vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, is $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{w}, \vec{v})$? *Ans:* No, usually not.

Example 5. Let $\vec{v} = (1, 2)$, $\vec{w} = (4, 1)$.

Q: Find a unit vector \vec{u} in the direction of \vec{w} .

Q: Draw a picture and use the above formula for dot products to explain why $\vec{v} \cdot \vec{u} = \|\text{proj}(\vec{v}, \vec{w})\|$.

Q: Find $\text{proj}(\vec{v}, \vec{w})$. *Ans:* $(\vec{v} \cdot \vec{u})\vec{u}$.

Theorem 2. $\text{proj}(\vec{v}, \vec{w}) = (\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}) \frac{\vec{w}}{\|\vec{w}\|}$

Proof. Denote $(\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}) (\frac{\vec{w}}{\|\vec{w}\|})$ by \vec{s} . By def of proj, we need to show \vec{s} is perp to $\vec{v} - \vec{s}$. So take their dot product: $\vec{s} \cdot (\vec{v} - \vec{s})$, plug in for \vec{s} , and simplify to show you get 0. □

So the scalar $\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}$ gives the *length* of "the shadow"!

Example 6. Let $\vec{v} = (1, 2)$. Let L be the line $y = x/4$. Find the vector \vec{p} on L that is closest to \vec{v} . But first, what does *closest* mean? It means the difference $\vec{v} - \vec{p}$ is as small as possible. Ans: The closest vector is $\text{proj}(\vec{v}, L)$. Our book denotes the difference $\vec{v} - \vec{p}$ by \vec{e} (for “error”).

The following theorem will be useful for problem 26 on page 183:

Theorem 3. For any square mtx A , the following statements are equivalent (i.e., they imply each other):

(1) A is invertible; (2) $\det(A) \neq 0$; (3) A has full col rank; (4) A has full row rank; (5) the cols of A are lin indep; (6) the rows of A are lin indep.
