Review:

Definition 1. Given two vectors $\vec{v} = (v_1, \cdots, v_n)$ and $\vec{w} = (w_1, \cdots, w_n)$, their **dot product** (also called **inner product**) is defined to be: $\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$.

Example 1. Let $\vec{u} = (1, 0)$, $\vec{v} = (3, 4)$, $\vec{w} = (-4, 3)$, $\vec{z} = (1/\sqrt{2}, 1/\sqrt{2})$, $\vec{x} = (1, 0, 1)$. Find each of the following:

$\vec{u} \cdot \vec{v}$ $\quad$ $\vec{w} \cdot \vec{v}$ $\quad$ $\vec{u} \cdot \vec{u}$ $\quad$ $\vec{v} \cdot \vec{v}$ $\quad$ $\vec{z} \cdot \vec{z}$ $\quad$ $\vec{x} \cdot \vec{x}$ $\quad$ $\vec{x} \cdot \vec{z}$

Ans: 3; 0; 1; 25; 1; 2; undefined.

Q: Is the dot product commutative? Ans: Y. Why?

Q: Is the dot product distributive w.r.t. vector addition? Ans: Y. Why?

Q: Is the dot product associative? Ans: N. The question doesn't really make sense. Why?

Q: Draw the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. Which ones are perpendicular to each other?

Q: Find the length of $\vec{v}$.

Q: Find the length of the vector $(1, 2, 3)$.

Definition 2. Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are said to be **perpendicular** or **orthogonal** to each other iff $\vec{v} \cdot \vec{w} = 0$; we write $\vec{v} \perp \vec{w}$. The **length** or **norm** of $\vec{v}$ is defined as $||\vec{v}|| = \sqrt{{v_1}^2 + \cdots + {v_n}^2} = \sqrt{\vec{v} \cdot \vec{v}}$. A **unit vector** is a vector whose length is 1.

Example 2. Q: Find the length of the vector $(1, 2, 3)/5$.

Q: Find a unit vector in the direction of $(1, 2, 3)$. Ans: Divide the vector $(1, 2, 3)$ by its length $||(1, 2, 3)|| = \sqrt{14}$; so unit vector $= (1, 2, 3)/\sqrt{14}$.

Q: What is the set of all vectors that are perpendicular to $(0, 0, 1)$? Ans: It is the plane passing through the origin, and perp to $(0, 0, 1)$; i.e., the xy-plane.

Q: Find two vectors that are perpendicular to $(1, 2, 3)$. Ans: We want (x, y, z) such that $x + 2y + 3z = 0$. So ...

Q: What is the set of all vectors that are perpendicular to $(1, 2, 3)$? Ans: It is the plane passing through the origin, and perp to $(1, 2, 3)$; it's equation is: $x + 2y + 3z = 0$.

Definition 3. A vector $\vec{u}$ is said to be **perpendicular** or **orthogonal** to a vector space V iff $\vec{u}$ is perpendicular to *every* vector in V; we write $\vec{u} \perp V$.

Definition 4. The **orthogonal complement** of a vector space $V \subset \mathbb{R}^n$ is defined as $V^\perp =$ the set of all vectors in $\mathbb{R}^n$ that are orthogonal to V.

Example 3. Q: In $\mathbb{R}^3$, let $V =$ the z-axis. What is $V^\perp$?

Q: In $\mathbb{R}^3$, what is the orthogonal complement of the xy-plane? Ans: the z-axis.

Q: In $\mathbb{R}^3$, are the xy-plane and the yz-plane orthogonal complements of each other? Ans: No, there are vectors in one plane that are not perp to the other plane.

Q: In $\mathbb{R}^4$ (with axes x_1, x_2, x_3, x_4), what is the orthogonal complement of the $x_1 x_2$-plane? Ans: the $x_3 x_4$-plane.

Theorem 1. (FTLA, part 2): For any $m \times n$ mtx A, its row space and nullspace are orthogonal complements of each other in $\mathbb{R}^n$; and its col space and left nullspace are orthogonal complements of each other in $\mathbb{R}^m$.

Example 4. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Find the four subspaces we associate to A. Ans: Here, col space and row space happen to be easier, so do those first: row space = xy-plane in $\mathbb{R}^3$; col space = xy-plane in $\mathbb{R}^2$. Now use FTLA to find nullspace and left nullspace: nullspace = z-axis; left nullspace = $\{0\}$.

Example 5. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Find the row space and the nullspace of A. Ans: row space = $\mathbb{R}^3$; so nullspace = $\{0\}$.

Let's see why the row space and the nullspace are orthogonal complements.

Step1. If $\vec{v} \in \mathrm{NS}(A)$, then $\vec{v}$ is perp to each row of A. Why? Use def of null space.

Step2. If $\vec{v} \in \mathrm{NS}(A)$, then $\vec{v}$ is perp to any lin comb of the rows of A. Why?

(By working with transposes, we see similarly that the col space and the left nullspace are also orthogonal.)

Discuss picture on the back cover of Strang's book.