Review defs: Span, Dim, Lin dep/indep.

Basis

Example 1. Let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Are $\vec{v_1}, \cdots, \vec{v_4}$ linearly independent? No. Why? How about $\vec{v_1}, \vec{v_2}, \vec{v_3}$? Yes. Why?

Q: Describe $W = \text{span}(\vec{v_1}, \cdots, \vec{v_4})$.

The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are lin indep and span W; we say they form a *basis* for W.

Definition 1. Let V be a vector space. A **basis** for V is a set of vectors $\vec{v_1}, \cdots, \vec{v_n}$ that (1) are linearly independent, and (2) span V.

Example 2. Q: Is the set $\{(1,0),(0,1)\}$ a basis for $\mathbb{R}^2$? Yes.

Q: Is the set $\{(1,0),(0,1),(2,3)\}$ a basis for $\mathbb{R}^2$? No, not lin indep.

Q: Is the set $\{(1,0,0),(0,1,0)\}$ a basis for $\mathbb{R}^3$? No; don't span.

Q: Is the set $\{(1,0,0),(0,1,1)\}$ a basis for $\mathbb{R}^3$? No. Why? Don't span.

Q: Give two different examples of bases for $\mathbb{R}^3$.

Example 3. Let V be the vector space spanned by: $\{(1,0,1),(1,1,0),(4,0,4),(2,1,1)\}$.

Q: What is the dimension of V?

Q: Is this vector space a line or a plane or neither? Plane.

Q: Find a basis for V.

Q: Must every basis for V have exactly two vectors in it? Yes, b/c of the following theorem.

Theorem 1. All bases of a vector space have the same number of vectors in them.

Proof: The book has a good proof; read if interested.

Note. Most books, including ours, define dim as follows:

Def: The **dimension** of a vector space is the number of vectors in a basis of that vector space.

Q: Is this def equivalent to our def? Yes; why? Ans: Let S be a set of fewest number of vectors necessary to span a vector space V. Then S is lin indep (why?). So S is a basis for V. So, by the theorem above, the number of vectors in S is the same as in any basis for V.

Note. Most books, including ours, give the following def for lin dep/indep, which looks different from but is equivalent to "our" def (except for the set $\{\vec{0}\}$; for this we agree to use the def below, which says it's lin dep).

Def: A set of vectors $\vec{v_1}, \cdots, \vec{v_n}$ is said to be **lin dep** if there exist scalars $c_1, \cdots, c_n$, at least one of which is non-zero, such that $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$. A set of vectors $\vec{v_1}, \cdots, \vec{v_n}$ is said to be **lin indep** if the only time $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$ is when all the scalars c_i are 0.

Examples ...

Row Space

Recall def of col space...

Q: Can you rephrase the def of col space using "span"? Ans: CS(A) = span(cols of A). So how would you define Row Space?

Definition 2. The **row space** of a matrix A is defined as: RS(A) = span(rows of A).

Example 4. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Q: What is the col space of A? $\mathbb{R}^2$. Why?

Q: What is the row space of A?

Q: CS(A) = RS(A)? No. Why?

Q: dim(CS(A)) = dim(RS(A))? Yes. Why?

Q: Is this a coincidence, that the row and col spaces have the same dim? Answer next time.

Example 5. Q: T or F: There exists a 2×3 mtx whose rows are lin indep. T. Why?

Q: T or F: There exists a 2×3 mtx whose rows are lin dep. T. Why?

Q: T or F: There exists a 2×3 mtx whose cols are lin dep. T. Why?

Q: T or F: There exists a 2×3 mtx whose cols are lin indep. F. Why? Because of the following thm.

Theorem 2. In an n-dimensional vector space, any set with more than n vectors is lin dep.

Proof: Skip.

HW # 18
Read sec 3.5.
p. 150: 2, 9, 16, 17ab, 22, 25abd.

Always prove or explain all your answers, even if the book doesn't ask for it!