Recall from last time:

After doing elimination (before back-substitution or back-elimination) the matrix is in *echelon* form (and upper-triangular form too? Yes).

Definition 1. A matrix is in **echelon form** iff:
1. under every pivot the entries are all zeros; and
2. the pivots are in a "staircase" arrangement, i.e., if p is a pivot, then every pivot to the right of p is below p (more precisely, for any two pivots $A_{i,j}$ and $A_{k,l}$, $i < k$ iff $j < l$); and
3. all rows of zeros appear at the bottom, i.e., below all pivots.

After Gaussian Elimination, if we continue with back-elimination, (i.e., we do Gauss-Jordan Elimination) we obtain a matrix in the following form.

Definition 2. A matrix is in **reduced row echelon form** (**rref**) iff:
1. it is in echelon form; and
2. the entries above each pivot are zero; and
3. all pivots equal 1.

The rref of a matrix A is denoted by rref(A).

Note. Our book sometimes just says "reduced form", instead of "reduced row echelon form."

Examples. Write various mtxs of various sizes, both in and not in rref.

Pivot Cols, Free Cols, Rank, Nullity

Example 1. Consider the following two systems of equations:

$$\text{System 1:}\quad \begin{aligned} x + 2y - z &= 3 \\ -2y + z &= 1 \end{aligned} \qquad \text{System 2:}\quad x + 2y - z = 3$$

Q: Find all sols for each system. How many sols does sys 1 have? How about sys 2? Ans: each sys has infinitely many sols.

But sys 2 has more sols. We have two free variables in sys 2, and only one free variable in sys 1.

The set of all sols to sys 2 is a 1 dimensional space.
The set of all sols to sys 1 is a 2 dimensional space.

We'll see what dimension means exactly next week. Today, we'll see names for the number of free vars, number of pivots, etc.

Definition 3. For a mtx in rref, the columns containing the pivots are called the **pivot columns**; the rest of the columns are called the **free columns**.

Definition 4. The **rank** of a mtx A is the number of pivot columns in rref(A). The **nullity** of a mtx A is the number of free cols in rref(A).

(The word nullity is not in our book.)

Examples:

Q: What are the rank and nullity of I_3? How about I_n? Ans: 3,0; n,0.

Q: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. What is rref($A$)? Ans: Identity matrix. Why? So what are the rank and nullity of A? Ans: 2 and 0.

Q: Is the rank of every $n \times n$ mtx $= n$? No. Why?

Q: Is the rank of every invertible $n \times n$ mtx $= n$? Yes. Why?

Q: True or false: for every $n \times n$ mtx, rank + nullity $= n$. Ans: T. Why?

Q: For every $m \times n$ mtx, rank + nullity =?. Ans: n. Why?

Q: Which gives us the number of the special sols for a mtx, rank or nullity? Ans: nullity.

Q: What is the maximum possible rank of an $m \times n$ mtx? Ans: the smaller of m and n. Why? b/c can have at most one pivot in each row, and in each col.

Recall: column space and nullspace are both vector spaces. Next week we'll learn the notion of dimension of a vec space. It turns out that
rank = dim of col space
nullity = dim of nullspace.

Theorem 1. For any $m \times n$ mtx A, $\text{rank}(A) = \text{rank}(A^T)$.

Proof not easy. Skip.

Q: T or F: For any $m \times n$ mtx A, $\text{nullity}(A) = \text{nullity}(A^T)$? False! Why?

HW # 15
Read sec 3.3. Preview sec 3.4.
p. 128: 1,3,8,9,12(a). p. 120: 15,22,23,27,28,29.

Always prove or explain all your answers, even if the book doesn't ask for it!