Today's outline:
1. Definition of the determinant
2. Properties of the det

Recall: Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A^{-1} exists iff $ad - bc \neq 0$.

Q: What does "iff" mean? It means each implies the other; either both true, or both false; if one is true, so is the other; if one is false, so is the other.

The number $ad - bc$ is called the *determinant* of A. Can you guess why its called that? Because it "determines" whether or not A is invertible.

Don't you wish there was such a number that played the same role for 3x3 mtxs? And all larger mtxs as well?
There is such a number. It's called the determinant. But computing it is not as staightforward as one would wish. :-(

<u>Notation</u>: The determinant of a mtx A is written as:

$$\det(A) \text{ or } |A|$$

In computing the det of a 3x3 mtx, we use det of 2x2 mtxs.
In computing the det of a 4x4 mtx, we use det of 3x3 mtxs.
And so on, for 5x5, 6x6, ...

Example 1. Let's compute the determinant of $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

$$\det \begin{bmatrix} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 1(5(2) - 1(1)) - 4(3(2) - 0(1)) + 2(3(1) - 0(5))$$

Explanation:
Step 1. Multiply each entry in the first row by the determinant of its "surviving" or "opposite" mtx (explained below).
Step 2. Alternately add and subtract.

Remark: Instead of the first row, could have used any other row or column. We'd get the same answer! We'll see more details later.

We need a precise definition.

Definition 1. (Not in book.) Let A be any mtx. The (i, j)-**minor** of A is the mtx obtained by removing its ith row and its jth column. It is denoted by $\hat{A}_{i,j}$. (Our book uses M_{ij}.)

Example: ...

Q: If A is a 5x5 mtx, then $\hat{A}_{2,3}$ is a ?x? mtx? Ans: 4x4.

Definition 2. Let A be any n by n mtx. The **determinant** of A is defined as:

If $n = 1$, then $\det(A) = A_{11}$.

If $n \geq 2$, then

$$\det(A) = A_{1,1} \det(\hat{A}_{1,1}) - A_{1,2} \det(\hat{A}_{1,2}) + \cdots + A_{1,n} \det(\hat{A}_{1,n})$$

More general definition: Fix any row i. Then,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j})$$

Or, fix any column j. Then,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j})$$

Compare this with the blue box on p. 223. Understand why they are the same!

Note. Although it may seem like it, this definition is not really a "circular" definition. It's a *recursive* definition.

Examples:
1x1 mtx: $\det([-5]) = ?$ Ans: -5 (not +5).
2x2 mtx: ...

det(zero-mtx)=? 0
det(I)=? 1

More examples: ... (use other rows or cols too, see that get same result)

Properties of the det

Theorem 1. A is invertivle iff $\det(A) \neq 0$.

Proof: Skip for now.

Theorem 2. $\det(AB) = \det(A)\det(B)$

Proof: Not very easy – will probably not have time to prove.

Corollary 3. $\det(A^{-1}) = 1/\det(A)$.

Proof. We know $AA^{-1} = I$. So take det of both sides, ..., then use above thm. □

HW #9
Read p. 223-4.
Do p. 226: 11(ignore "cofactor"); p. 213: 2, 3, 12, 27acd. Ch: p.226: 14.

Always prove or explain all your answers, even if the book doesn't ask for it!