Equivalent systems

Definition 1. Two systems of linear equations are said to be **equivalent** if they have the same solutions (i.e., if any solution of one system is also a solution of the other).

Example 1. Are the following systems equiv? Why or why not?

$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 5 \end{aligned} \qquad \text{vs.} \qquad 4x + y = 9$$

(eq1 + eq2 in system1 gives the eq in system2)
Ans: No. The first system has one sol: $x = 2$, $y = 1$. The second system has a line of sols.

Example 2. How about the following two systems, are they equiv?

$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 5 \end{aligned} \qquad \text{vs.} \qquad \begin{aligned} x + 2y &= 4 \\ 6x - 2y &= 10 \end{aligned}$$

Ans: Yes. Proof: If $x = a$ and $y = b$ satisfy both eqs of sys1, then they must also satisfy both eqs of sys2, and vice versa.

Example 3. How about the following two systems, are they equiv?

$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 5 \end{aligned} \qquad \text{vs.} \qquad \begin{aligned} x + 2y &= 4 \\ 4x + y &= 9 \end{aligned}$$

Ans: Yes. Proof: If $x = a$ and $y = b$ satisfy both eqns of sys1, then they must also satisfy eq1 of sys2 (same as before), and eq2 of sys2 because eq2sys2 = eq1sy1 + eq2sys1.
Is this enough? No. Must also prove that if $x = a$ and $y = b$ satisfy both eqns of sys2, then they must satisfy both eqns of sys1.
How do we show this? We need to derive eq2sys1 from the eqns of sys2: eq2sys1 = eq2sys2 - eq1sys2.
(Pictorially, each system corresponds to a different pair of lines, but with the same intersection point!)

Conclusion: If we can **derive** each system from the other, then they are equivalent.

Example 4. How about the following two systems, are they equiv?

$$\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 8 \end{aligned} \qquad \text{vs.} \qquad \begin{aligned} -x - y &= -3 \\ x + y &= 3 \end{aligned}$$

(We did the following two operations on the first system SIMULTANEOUSLY to get the second system: eq1 = eq1 - eq2 ; eq2 = eq2 - eq1.)

Ans: No. The original system has only one solution: $x = 1$, $y = 2$. But the new system has other solutions as well, such as $x = 5$, $y = -2$.

What's the lesson we learn from all this? Two systems are equiv if we can derive each one from the other. But how do you tell whether or not each can be derived from the other? I can't think of any *simple*, systematic, and general method. Can you?

So, to be safe, we will just stick to the systematic Gaussian Elimination method.

Free variables

Example 5. Solve the following system: $\begin{aligned} x+y+z&=1\\ x+2y+z&=3 \end{aligned}$

Do: eq2=eq2-eq1; get: $y=2$. But can't find x and z.

Q: Does this mean there is no solution? No. There are infinitely many! We designate z as a **free variable**. Then $x=-1-z$. So the set of all solutions is written as: $\{(x,y,z) \mid x=1-z, \quad y=2, \quad z \in \mathbb{R}\}$. (We could have also picked x as the free variable. Then we would have gotten $z=-1-x$.)

Row operations

Definition 2. A **row operation** is any of the following, where c is a nonzero scalar:
1. row$i$ = row$i$ + c(row$k$);
2. row$i$ = c(row$i$);
3. row$i$ = row$k$, and row$k$ = row$i$ (switch rows).

Definition 3. A **pivot** is the first non-zero entry in a row.

Outline of Gaussian Elimination:
1. Find leftmost pivot.
2. If necessary, do row-exchange to "bring it up".
3. (Optional) Divide to make pivot = 1.
4. Make zeros under pivot.
5. Find next leftmost pivot.
6. Go to step 2.

Example 6. Solve the following system using Gaussian Elimination: $\begin{aligned} y+z&=5\\ 2x+6y-2z&=8\\ x+4y+z&=9 \end{aligned}$.

$$\begin{bmatrix} 0 & 1 & 1 & \mid & 5\\ 2 & 6 & -2 & \mid & 8\\ 1 & 4 & 1 & \mid & 9 \end{bmatrix} \to \cdots$$

Operations:
1. Switch row1 and row 2 to bring up leftmost pivot.
2. Make pivot =1.
3. Make zeros under pivot.
4. Next leftmost pivot is at row2, col2. Pivot already =1. Make zeros under it.

We finally get: $\begin{bmatrix} 1 & 3 & -1 & \mid & 4\\ 0 & 1 & 1 & \mid & 5\\ 0 & 0 & 1 & \mid & 0 \end{bmatrix}$

HW #4, due Fri 31 Jan
Read sec 2.3. Preview section 2.4.
Do p. 52: 23-26.