FOR HW 27

Recall "Main Theorem" from class:

Theorem 1. Let A be an $m \times n$ matrix. Then
(a) $(\text{row}(A))^\perp = \text{null}(A)$;
(b) $(\text{null}(A))^\perp = \text{row}(A)$.

Proof: Let $R = \text{row}(A)$, $N = \text{null}(A)$.

We proved part (a) in class, in two steps (you don't need to reprove them here; just use them in the proof of part (b)):

Step 1. We showed that every $\vec{r} \in R$ is orthogonal to every $\vec{n} \in N$. It follows from this and the definition of "orthogonal complement" that $N \subseteq R^\perp$.

Step 2. We showed $R^\perp \subseteq N$.

Steps 1 and 2 prove part (a).

Problem 1. To prove part (b), first prove the following lemma (note: lemma = theorem; the only difference is: "lemma" is usually used to mean a "helping theorem", i.e., a theorem that we use in the proof of another "more important" theorem).

Lemma 2. Let A be an $m \times n$ matrix. Then every vector $\vec{v} \in \mathbb{R}^n$ can be written as $\vec{v} = \vec{r} + \vec{n}$ for some $\vec{r} \in \text{row}(A)$ and $\vec{n} \in \text{null}(A)$.

Hint: Let $\{\vec{r_1}, \cdots, \vec{r_k}\}$ be an orthogonal basis for R. Let $\{\vec{n_1}, \cdots, \vec{n_l}\}$ be an orthogonal basis for N. Prove that $\{\vec{r_1}, \cdots, \vec{r_k}, \vec{n_1}, \cdots, \vec{n_l}\}$ is linearly independent and spans $\mathbb{R}^n$.

Problem 2. Now prove part (b) of the Main Theorem. First use Step 1 above to show $R \subseteq N^\perp$. Then show $N^\perp \subseteq R$. Hint: Let $\vec{v} \in N^\perp$. We want to show $\vec{v} \in R$. By the above lemma, $\vec{v} = \vec{r} + \vec{n}$ for some ... . So $\vec{v} \cdot \vec{n} = (\vec{r} + \vec{n}) \cdot \vec{n}$. So ... . So $\vec{v} = \vec{r}$.

Problem 3. Prove the following lemma without using the Orthogonal Decomposition Theorem.

Lemma 3. Let A be an $m \times n$ matrix. Then every vector $\vec{v} \in \mathbb{R}^n$ can be written as $\vec{v} = \vec{r} + \vec{n}$ for a unique $\vec{r} \in \text{row}(A)$ and a unique $\vec{n} \in \text{null}(A)$.

Hint: Suppose $\vec{v} = \vec{r} + \vec{n}$ and $\vec{v} = \vec{r'} + \vec{n'}$. We want to show $\vec{r} = \vec{r'}$ and $\vec{n} = \vec{n'}$. (1) Show $\vec{r} - \vec{r'} \in N$. (2) Show $\vec{r} - \vec{r'} = \vec{0}$ by using Theorem 5.9(c).

Problem 4. Use the above lemmas and theorems to prove the Orthogonal Decomposition Theorem. Hint: Given a subspace W, show there is a matrix A whose row space equals W.