Examples of applications of matrix multiplication (not in book)

Example 1. We have two bags of trail-mix, each containing almonds and raisins. Bag 1: 90% almonds, 10% raisins. Bag 2: 20% almonds, 80% raisins. Let's denote this by the matrix $A = \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}$. Suppose almonds contain 60% protein, 40% fat and carbohydrates, while raisins contain 0% protein, 100% fat and carbohydrates; Let's denote this by $B = \begin{bmatrix} .6 & .4 \\ 0 & 1 \end{bmatrix}$.

Q: Compute AB. What does each entry of AB represent? A: The matrix AB gives the protein vs. fat/carb content of each bag.

Q: Suppose we want to make a third bag containing almonds and raisin 50-50. What percentage of bag 3 should come from each of the two bags? A: Let x_i represent the percentage of bag 3 that comes from bag i. Then: $\text{row}_1(A)x_1 + \text{row}_2(A)x_2 = [.5 \;\; .5]$, i.e., $A^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A^T)^{-1} \begin{bmatrix} .5 \\ .5 \end{bmatrix}$.

Example 2. Rotation Matrices. Let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$. Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, where α is some arbitrary angle in radians. Any 2×2 matrix that can be written in this form is called a **rotation matrix**.

Q: Let $\vec{w} = A\vec{v}$. How is $\vec{w}$ related to $\vec{v}$? A: $\vec{w}$ is obtained by rotating $\vec{v}$ counterclockwise through an angle of α.

Proof. We can write x and y as: $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then $A\vec{v} = \begin{bmatrix} \cos(\alpha)r\cos(\theta) - \sin(\alpha)r\sin(\theta) \\ \sin(\alpha)r\cos(\theta) + \cos(\alpha)r\sin(\theta) \end{bmatrix} = \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$, as desired. □

Q: Let $B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$, where β is also some arbitrary angle. Is AB a rotation matrix? A: Yes. Why?

Q: What is A^{-1}? Is it a rotation matrix of angle $-\alpha$? A: Yes. Why?

Example 3. (From *Elementary Linear Algebra*, by Spence, Insel, and Friedberg, page 108.) We have four cities with airports. Let A be the 4×4 matrix defined by

$$A_{i,j} = \begin{cases} 1 & \text{if there is a nonstop flight from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

Q: Suppose $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ Compute A^2. What does $A^2_{2,3}$ represent? A: It represents the number of flights with exactly one layover between city 2 and city 3. Why?

(Can we do more? Does A^3 represent the flights with exactly two layovers? Why?)
