

Definition 1. Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors. To say a vector \vec{w} is a **linear combination** of $\vec{v}_1, \dots, \vec{v}_n$ means there exist scalars $c_1, \dots, c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{w}$. The numbers c_1, \dots, c_n are called **coefficients**.

Definition 2. A polynomial in several variables is said to be **linear** if the exponent of every variable is 1, and each variable is multiplied only by a constant, not by other variables.

Definition 3. A system is called **singular** if it has no solutions or infinitely many solutions. A system is called **nonsingular** if it has exactly one solution.

Definition 4. Two systems of linear equations are said to be **equivalent** if they have the same solutions (i.e., if any solution of one system is also a solution of the other).

Definition 5. A **row operation** is any of the following, where c is a nonzero scalar: 1. $\text{row } i = \text{row } i + c(\text{row } k)$; 2. $\text{row } i = c(\text{row } i)$; 3. $\text{row } i = \text{row } k$, and $\text{row } k = \text{row } i$ (switch rows).

Definition 6. A **pivot** is the first non-zero entry in a row.

Definition 7. Let A be an $m \times n$ matrix (m rows, n columns). If $m = n$, then A is said to be a **square matrix**. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the (i, j) -**entry** of A , denoted by $A_{i,j}$, is the number in the i th row and the j th column of A . We denote the i th row of A by **row** $_i(A)$, and the j th column of A by **col** $_j(A)$.

Definition 8. Let A and B be $m \times n$ matrices. Then their **sum** $A + B$ is an $m \times n$ matrix C defined by: $C_{i,j} = A_{i,j} + B_{i,j}$.

Definition 9. Let A be an $m \times n$ matrix, and B an $n \times q$ matrix. Then their **product** AB is an $m \times q$ matrix C defined by $C_{i,j} = \text{row}_i(A) \text{col}_j(B)$. (Equivalently, C can be defined by: $\text{col}_j(C) = A \text{col}_j(B)$.)

Definition 10. The $n \times n$ **identity matrix** I_n is defined to have 1's along its diagonal, and 0's elsewhere.

Definition 11. Two $n \times n$ matrices A and B are said to be **inverses** of each other if $AB = I_n$ and $BA = I_n$.

Definition 12. A matrix A that has an inverse is called **invertible**, and its inverse is denoted by A^{-1} . A matrix that does not have an inverse is called **non-invertible** or **singular**.

Definition 13. The **transpose** A^T of a matrix A is defined by: the ij th entry of A^T is the ji th entry of A ; i.e., $(A^T)_{i,j} = A_{j,i}$

Definition 14. A matrix A is called **symmetric** if $A^T = A$.

Definition 15. Let A be any matrix. The (i, j) -**minor** of A is the matrix obtained by removing its i th row and its j th column. It is denoted by $\hat{A}_{i,j}$. (Our book uses M_{ij} .)

Definition 16. Let A be any n by n matrix. The **determinant** of A is defined as: If $n = 1$, then $\det(A) = A_{11}$. If $n \geq 2$, then for any fixed row i ,

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j});$$
 or, for any fixed col-

$$\text{umn } j, \det(A) = \sum_{i=1}^n (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j}).$$

Definition 17. Given a matrix A , the **cofactor** of its (i, j) -entry is defined as $C_{i,j} = (-1)^{i+j} \det(\hat{A}_{i,j})$.

Definition 18. Let \vec{u} and \vec{v} be vectors in \mathbb{R}^3 . Their **cross product** is defined as $\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$.

Definition 19. The **column space** of an $m \times n$ matrix A is the set (i.e., the collection) of all vectors that are lin combs of the columns of A .

Definition 20. Let V be a set of vectors in \mathbb{R}^n (for some n). V is said to be a **vector space** if it satisfies both of the following conditions: (1) It is **closed under vector addition**: for every \vec{v} and \vec{w} in V , $\vec{v} + \vec{w}$ is in V . (2) It is **closed under scalar multiplication**: for every \vec{v} in V and for every scalar c , $c\vec{v}$ is in V .

Definition 21. If V and W are both vector spaces, and if $V \subset W$, then we say V is a **subspace** of W . (" \subset " is the subset symbol; it means: W contains all the vectors that are in V , plus more perhaps.)

Definition 22. Given any matrix A , the set of all vectors \vec{x} that satisfy $A\vec{x} = \vec{0}$ is called the **nullspace** of A , denoted $\text{NS}(A)$. (Book uses boldface N.)

Definition 23. The **trivial vector space** is the set $\{\vec{0}\}$ (it contains only the zero vector).

Definition 24. A matrix is in **echelon form** iff: 1. under every pivot the entries are all zeros; and 2. the pivots are in a "staircase" arrangement, i.e., if p is a pivot, then every pivot to the right of p is below p (more precisely, for any two pivots $A_{i,j}$ and $A_{k,l}$, $i < k$ iff $j < l$); and 3. all rows of zeros appear at the bottom, i.e., below all pivots.

Definition 25. A matrix is in **reduced row echelon form (rref)** iff: 1. it is in echelon form; and 2. the entries above each pivot are zero; and 3. all pivots equal 1.

Definition 26. The **rank** of a mtx A is the number of pivot columns in $\text{rref}(A)$. The **nullity** of a mtx A is the number of free cols in $\text{rref}(A)$.

Definition 27. In $A\vec{x} = \vec{b}$, if $\vec{b} = \vec{0}$, we say the system is **homogeneous**. Otherwise we say it is **nonhomogeneous**.

Definition 28. An $m \times n$ mtx A is said to have **full column rank** iff its rank equals its number of columns ($r=n$); i.e., there is a pivot in every col. A is said to have **full row rank** iff its rank equals its number of rows ($r=m$); i.e., there is a pivot in every row.

Definition 29. Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors in \mathbb{R}^k . Their **span** is defined as:

$\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \{\vec{w} \mid \vec{w} \text{ is a lin comb of } \vec{v}_1, \dots, \vec{v}_n\}$
= the set of all vectors that are linear combinations of $\vec{v}_1, \dots, \vec{v}_n$.

Definition 30. For any vector space V , the **dimension** of V , denoted $\dim(V)$, is the least number of vectors necessary to span V .

Definition 31. A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is said to be **linearly dependent** iff at least one of them is equal to a linear combination of the others. A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is said to be **linearly independent** iff none of them is equal to a linear combination of the others.

Equivalent Def: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is said to be **linearly dependent** if there exist scalars c_1, \dots, c_n , at least one of which is nonzero, such that $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$. The vectors $\vec{v}_1, \dots, \vec{v}_n$ are said to be **linearly independent** if the only time $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$ is when all the scalars c_i are 0.

Definition 32. Given a vector space V , a **basis** for V is a set of vectors $\vec{v}_1, \dots, \vec{v}_n$ that (1) are linearly independent, and (2) span V .

Definition 33. The **row space** of a matrix A is defined as: $\text{RS}(A) = \text{span}(\text{rows of } A)$.

Definition 34. The **left nullspace** of an $m \times n$ mtx A is the set of all $1 \times m$ row vectors \vec{y} such that $\vec{y}A = \vec{0}$. We write: $\text{LNS}(A) = \{\vec{y} \mid \vec{y}A = \vec{0}\}$. Equivalent definition: $\text{LNS}(A) = \text{NS}(A^T)$.

Definition 35. Given two vectors $\vec{v} = (v_1, \dots, v_n)$ and $\vec{w} = (w_1, \dots, w_n)$, their **dot product** (also called **inner product**) is defined to be: $\vec{v} \cdot \vec{w} = v_1w_1 + \dots + v_nw_n$.

Definition 36. Two vectors \vec{v} and \vec{w} in \mathbb{R}^n are said to be **perpendicular** or **orthogonal** to each other iff $\vec{v} \cdot \vec{w} = 0$; we write $\vec{v} \perp \vec{w}$. The **length** or **norm** of \vec{v} is defined as $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}$. A **unit vector** is a vector whose length is 1.

Definition 37. A vector \vec{u} is said to be **perpendicular** or **orthogonal** to a vector space V iff \vec{u} is perpendicular to every vector in V ; we write $\vec{u} \perp V$.

Definition 38. The **orthogonal complement** of a vector space $V \subset \mathbb{R}^n$ is defined as $V^\perp =$ the set of all vectors in \mathbb{R}^n that are orthogonal to V .

Definition 39. For $\vec{v} \in \mathbb{R}^n$ and W a subspace of \mathbb{R}^n , the **projection** of \vec{v} onto W is defined as: $\text{proj}(\vec{v}, W) =$ a vector $\vec{p} \in W$ such that $\vec{v} - \vec{p}$ is orthogonal to W .

Definition 40. For $\vec{v}, \vec{w} \in \mathbb{R}^n$, the **projection** of \vec{v} onto \vec{w} is defined as: $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{v}, L)$, where L is the line that contains \vec{w} .

Definition 41. A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is said to be **orthonormal** if every \vec{v}_i is a unit vector, and the vectors are all mutually perpendicular.

Definition 42. Let A be a square matrix. A nonzero vector \vec{v} is an **eigenvector** for A if $A\vec{v}$ has the same direction as $\pm\vec{v}$, i.e., $A\vec{v} = \lambda\vec{v}$ for some scalar λ ; λ

is said to be the **eigenvalue** that corresponds to the eigenvector \vec{v} .

Definition 43. Given a square matrix A , $\det(A - \lambda I)$ is called the **characteristic polynomial** of A .

Definition 44. A matrix A is said to be **diagonal** if its nondiagonal entries are all zero, i.e., for $i \neq j$, $A_{i,j} = 0$.

Definition 45. A matrix A is said to be **diagonalizable** if there exists a matrix B such that $B^{-1}AB$ is a diagonal matrix.

Theorems

Theorem 1. Matrix multiplication is **associative**: $(AB)C = A(BC)$ (assuming “sizes match”, so that the products are defined).

Theorem 2. Matrix multiplication is **distributive** with respect to addition: $A(B + C) = AB + AC$ (assuming “sizes match”, so that the products are defined).

Theorem 3. If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 4. If $ad - bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Theorem 5. $(A + B)^T = A^T + B^T$. $(AB)^T = B^T A^T$.

Corollary 6. If A is invertible, then $(A^{-1})^T = (A^T)^{-1}$.

Theorem 7. A is invertible iff $\det(A) \neq 0$.

Theorem 8. $\det(AB) = \det(A)\det(B)$

Corollary 9. If A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

Theorem 10. For any square matrix A , $\det(A^T) = \det(A)$.

Theorem 11. (1) *Constant Multiple Rule:* If B is obtained by multiplying one row or one column of a square matrix A by a constant c , then $\det(B) = c\det(A)$.

(2) *Switching Rule:* Switching any two rows of a matrix changes its det by a factor of -1 . (Works also for nonadjacent rows.) Switching any two columns of a matrix changes its det by a factor of -1 . (Works also for nonadjacent cols.)

(3) *Adding Rule:* Adding a multiple of one row to another (or of one column to another) does not change the det.

Theorem 12. The area of the parallelogram with sides $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$ is given by the absolute value of the det of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Theorem 13. Given three vectors $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$, the area of the parallelepiped in \mathbb{R}^3 with these vectors as its sides is equal to the absolute value of the det of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$.

Theorem 14. For any $m \times n$ mtix A , $\text{rank}(A) = \text{rank}(A^T)$.

Theorem 15. All bases of a vector space have the same number of vectors in them.

Theorem 16. In an n -dimensional vector space, any set with more than n vectors is lin dep.

Theorem 17. For every matrix A , each of $\text{CS}(A)$, $\text{RS}(A)$, $\text{NS}(A)$, and $\text{LNS}(A)$ is a vector space.

Theorem 18. Fundamental Theorem of Linear Algebra, Part I (FTLA): For any $m \times n$ mtix A , $\dim(\text{RS}(A)) + \dim(\text{NS}(A)) = n$; $\dim(\text{CS}(A)) + \dim(\text{LNS}(A)) = m$.

Theorem 19. Column operations do not change the column space of a matrix. Row operations do not change the row space of a matrix. Row operations do not change the nullspace of a matrix.

Theorem 20. Row operations on a matrix do not change the lin dep/indep of the cols. Col operations on a matrix do not change the lin dep/indep of the rows.

Theorem 21. For any $m \times n$ mtix A , the pivot cols form a basis for $\text{CS}(A)$. So $\dim(\text{CS}(A)) = \text{rank}(A)$.

Note. By pivot cols of A we mean those columns that, after Gauss-Jordan elimination, become pivot cols in $\text{rref}(A)$.

Theorem 22. For any $m \times n$ mtix A , the special sols to $A\vec{x} = \vec{0}$ form a basis for $\text{NS}(A)$. Therefore $\dim(\text{NS}(A)) = \text{nullity}(A)$.

Theorem 23. For any matrix A , $\dim(\text{CS}(A)) = \dim(\text{RS}(A))$. In other words, for any mtix A , # of lin indep cols = # of lin indep rows!

Theorem 24. (FTLA, part 2): For any $m \times n$ mtix A , its row space and nullspace are orthogonal complements of each other in \mathbb{R}^n ; and its col space and left nullspace are orthogonal complements of each other in \mathbb{R}^m .

Theorem 25. Given any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)$, where θ is the angle between the two vectors.

Theorem 26. $\text{proj}(\vec{v}, \vec{w}) = (\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}) \frac{\vec{w}}{\|\vec{w}\|}$

Theorem 27. For any square mtix A , the following statements are equivalent (i.e., they imply each other): (1) A is invertible; (2) $\det(A) \neq 0$; (3) A has full col rank; (4) A has full row rank; (5) the cols of A are lin indep; (6) the rows of A are lin indep.

Theorem 28. Every orthonormal set is linearly independent.

Theorem 29. In an n -dimensional vector space W , a set of n vectors is lin indep iff it spans W .

Theorem 30. Suppose A is an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues (not necessarily distinct). Let Λ be the diagonal matrix with $\Lambda_{i,i} = \lambda_i$. Let S be the matrix whose j th column is the eigenvector \vec{v}_j . Then $AS = S\Lambda$; so $S^{-1}AS = \Lambda$.

Theorem 31. (Converse of previous theorem.) Let A and B be $n \times n$ matrices such that $B^{-1}AB$ is a diagonal matrix D . Then each column of B is an eigenvector of A , with corresponding eigenvalues appearing on the diagonal of D .

Theorem 32. An $n \times n$ mtix is diagonalizable iff it has n lin indep e-vecs.
