Recall def of dot product.

A nice and useful interpretation of the dot product

Theorem 1. Given any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cos(\theta)$, where θ is the angle between the two vectors.

Sketch of Proof in $\mathbb{R}^2$ (draw picture): Let $r = ||\vec{v}||$, $s = ||\vec{w}||$. Then $\vec{v} = (r\cos(\alpha), r\sin(\alpha))$, and $\vec{w} = (s\cos(\beta), s\sin(\beta))$. So $\vec{v} \cdot \vec{w} = r\cos(\alpha)s\cos(\beta) + r\sin(\alpha)s\sin(\beta)$, which by a trig identity, equals $rs\cos(\alpha - \beta)$. QED.

Example 1. Use the above theorem to find the angle between $\vec{v} = (1, 2, 3)$ and $\vec{w} = (2, 0, -1)$.

Recall def of orthogonal: $\vec{v} \perp \vec{w}$, $\vec{v} \perp V$.

Example 2. Q: Let $\vec{v} = (1, 2) \in \mathbb{R}^2$. We shine light from above perpendicular to the x-axis. Find the shadow of $\vec{v}$ as a vector on the x-axis. Ans: The shadow is $\vec{s} = (1, 0)$.

Q: Let $\vec{v} = (1, 2) \in \mathbb{R}^3$. We shine light perpendicular to the line L given by the eqn $y = x/2$. Let's denote the shadow of $\vec{v}$ onto L by $\vec{s}$. Draw a picture, showing the vectors $\vec{v}$, $\vec{s}$, $\vec{v} - \vec{s}$. What is the angle between $\vec{s}$ and $\vec{v} - \vec{s}$? Ans: 90 degrees. Why? Because the light is perp to the line.

We use the above observation to give a precise def of "shadow." Instead of "shadow" we say *projection*.

Definition 1. For $\vec{v} \in \mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$, the **projection** of $\vec{v}$ onto W is defined as: $\text{proj}(\vec{v}, W)$ = a vector $\vec{p} \in W$ such that $\vec{v} - \vec{p}$ is orthogonal to W.

Note. The notation "proj" is not in our book.

Example 3. Let $\vec{v} = (1, 2, 3)$, $W = xy$-plane. Find $\text{proj}(\vec{v}, W)$. Ans: $(1, 2, 0)$.

Definition 2. For $\vec{v}, \vec{w} \in \mathbb{R}^n$, the **projection** of $\vec{v}$ onto $\vec{w}$ is defined as: $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{v}, L)$, where L is the line that contains $\vec{w}$.

Example 4. Let $\vec{v} = (1, 2)$, $\vec{w} = (0, 5)$. Find $\text{proj}(\vec{v}, \vec{w})$. Ans: $L = y$-axis, so $\text{proj}(\vec{v}, \vec{w}) = (0, 2)$.

Q: In general, given arbitrary vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, is $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{v}, 2\vec{w})$? Ans: Yes; $\vec{w}$ and $2\vec{w}$ give the same line L.

Q: In general, given arbitrary vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, is $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{w}, \vec{v})$? Ans: No, usually not.

Example 5. Let $\vec{v} = (1, 2)$, $\vec{w} = (4, 1)$.

Q: Find a unit vector $\vec{u}$ in the direction of $\vec{w}$.

Q: Draw a picture and use the above formula for dot products to explain why $\vec{v} \cdot \vec{u} = ||\text{proj}(\vec{v}, \vec{u})||$.

Q: Find $\text{proj}(\vec{v}, \vec{w})$. Ans: $(\vec{v} \cdot \vec{u})\vec{u}$.

Theorem 2. $\text{proj}(\vec{v}, \vec{w}) = (\vec{v} \cdot \frac{\vec{w}}{||\vec{w}||})\frac{\vec{w}}{||\vec{w}||}$

Proof. Denote $(\vec{v} \cdot \frac{\vec{w}}{||\vec{w}||})(\frac{\vec{w}}{||\vec{w}||})$ by $\vec{s}$. By def of proj, we need to show $\vec{s}$ is perp to $\vec{v} - \vec{s}$. So take their dot product: $\vec{s} \cdot (\vec{v} - \vec{s})$, plug in for $\vec{s}$, and simplify to show you get 0. □

So the scalar $\vec{v} \cdot \frac{\vec{w}}{||\vec{w}||}$ gives the *length* of "the shadow"!

Example 6. Let $\vec{v} = (1, 2)$. Let L be the line $y = x/4$. Find the vector $\vec{p}$ on L that is closest to $\vec{v}$. But first, what does *closest* mean? It means the difference $\vec{v} - \vec{p}$ is as small as possible. Ans: The closest vector is $\text{proj}(\vec{v}, L)$. Our book denotes the difference $\vec{v} - \vec{p}$ by $\vec{e}$ (for "error").

The following theorem will be useful for problem 26 on page 183:

Theorem 3. For any square mtx A, the following statements are equivalent (i.e., they imply each other):

(1) A is invertible; (2) $\det(A) \neq 0$; (3) A has full col rank; (4) A has full row rank; (5) the cols of A are lin indep; (6) the rows of A are lin indep.

HW # 23, due Mon 19 Nov

Preview sec 4.4. Do: p. 17: 5,6c,11,12.