

5+7-Nov-2001

---

Review:

*Definition 1.* Given two vectors  $\vec{v} = (v_1, \dots, v_n)$  and  $\vec{w} = (w_1, \dots, w_n)$ , their **dot product** (also called **inner product**) is defined to be:  $\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n$ .

*Example 1.* Let  $\vec{u} = (1, 0)$ ,  $\vec{v} = (3, 4)$ ,  $\vec{w} = (-4, 3)$ ,  $\vec{z} = (1/\sqrt{2}, 1/\sqrt{2})$ ,  $\vec{x} = (1, 0, 1)$ . Find each of the following:

$\vec{u} \cdot \vec{v}$        $\vec{w} \cdot \vec{v}$        $\vec{u} \cdot \vec{u}$        $\vec{v} \cdot \vec{v}$        $\vec{z} \cdot \vec{z}$        $\vec{x} \cdot \vec{x}$        $\vec{x} \cdot \vec{z}$

Ans: 3; 0; 1; 25; 1; 2; undefined.

Q: Is the dot product commutative? Ans: Y. Why?

Q: Is the dot product distributive w.r.t. vector addition? Ans: Y. Why?

Q: Is the dot product associative? Ans: N. The question doesn't really make sense. Why?

Q: Draw the vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ . Which ones are perpendicular to each other?

Q: Find the length of  $\vec{v}$ .

Q: Find the length of the vector  $(1, 2, 3)$ .

*Definition 2.* Two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are said to be **perpendicular** or **orthogonal** to each other iff  $\vec{v} \cdot \vec{w} = 0$ ; we write  $\vec{v} \perp \vec{w}$ . The **length** or **norm** of  $\vec{v}$  is defined as  $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}$ . A **unit vector** is a vector whose length is 1.

*Example 2.* Q: Find the length of the vector  $(1, 2, 3)/5$ .

Q: Find a unit vector in the direction of  $(1, 2, 3)$ . Ans: Divide the vector  $(1, 2, 3)$  by its length  $\|(1, 2, 3)\| = \sqrt{14}$ ; so unit vector =  $(1, 2, 3)/\sqrt{14}$ .

Q: What is the set of all vectors that are perpendicular to  $(0, 0, 1)$ ? Ans: It is the plane passing through the origin, and perp to  $(0, 0, 1)$ ; i.e., the  $xy$ -plane.

Q: Find two vectors that are perpendicular to  $(1, 2, 3)$ . Ans: We want  $(x, y, z)$  such that  $x + 2y + 3z = 0$ . So ...

Q: What is the set of all vectors that are perpendicular to  $(1, 2, 3)$ ? Ans: It is the plane passing through the origin, and perp to  $(1, 2, 3)$ ; its equation is:  $x + 2y + 3z = 0$ .

*Definition 3.* A vector  $\vec{u}$  is said to be **perpendicular** or **orthogonal** to a vector space  $V$  iff  $\vec{u}$  is perpendicular to *every* vector in  $V$ ; we write  $\vec{u} \perp V$ .

*Definition 4.* The **orthogonal complement** of a vector space  $V \subset \mathbb{R}^n$  is defined as  $V^\perp =$  the set of all vectors in  $\mathbb{R}^n$  that are orthogonal to  $V$ .

*Example 3.* Q: In  $\mathbb{R}^3$ , let  $V =$  the  $z$ -axis. What is  $V^\perp$ ?

Q: In  $\mathbb{R}^3$ , what is the orthogonal complement of the  $xy$ -plane? Ans: the  $z$ -axis.

Q: In  $\mathbb{R}^3$ , are the  $xy$ -plane and the  $yz$ -plane orthogonal complements of each other? Ans: No, there are vectors in one plane that are not perp to the other plane.

Q: In  $\mathbb{R}^4$  (with axes  $x_1, x_2, x_3, x_4$ ), what is the orthogonal complement of the  $x_1 x_2$ -plane? Ans: the  $x_3 x_4$ -plane.

*Theorem 1.* (FTLA, part 2): For any  $m \times n$  mtx  $A$ , its row space and nullspace are orthogonal complements of each other in  $\mathbb{R}^n$ ; and its col space and left nullspace are orthogonal complements of each other in  $\mathbb{R}^m$ .

---

*Example 4.* Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

Find the four subspaces we associate to  $A$ . Ans: Here, col space and row space happen to be easier, so do those first: row space =  $xy$ -plane in  $\mathbb{R}^3$ ; col space =  $xy$ -plane in  $\mathbb{R}^2$ . Now use FTLA to find nullspace and left nullspace: nullspace =  $z$ -axis; left nullspace =  $\{0\}$ .

---

*Example 5.* Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

Find the row space and the nullspace of  $A$ . Ans: row space =  $\mathbb{R}^3$ ; so nullspace =  $\{0\}$ .

---

Let's see why the row space and the nullspace are orthogonal complements.

Step1. If  $\vec{v} \in \text{NS}(A)$ , then  $\vec{v}$  is perp to each row of  $A$ . Why? Use def of null space.

Step2. If  $\vec{v} \in \text{NS}(A)$ , then  $\vec{v}$  is perp to any lin comb of the rows of  $A$ . Why?

(By working with transposes, we see similarly that the col space and the left nullspace are also orthogonal.)

---

Discuss picture on the back cover of Strang's book.

---

---

---

**HW # 21, due Fri 9 Nov**

Read sec 1.2. Do: p. 17: [1,3,6,10],4,8ab,16.

**HW # 22, due Mon 12 Nov**

Read sec 4.1. Do: p. 171: 5,13,15-19,25.

Always prove or explain all your answers, even if the book doesn't ask for it!