Review from last time: Def of left null space; theorems:

Theorem Fundamental Theorem of Linear Algebra, Part I (FTLA): For any $m \times n$ mtx A,

$$\dim(\mathrm{RS}(A)) + \dim(\mathrm{NS}(A)) = n$$

$$\dim(\mathrm{CS}(A)) + \dim(\mathrm{LNS}(A)) = m$$

Theorem Column operations do not change the column space of a matrix. Row operations do not change the row space of a matrix.

Theorem Row operations on a matrix do not change the lin dep/indep of the cols. Col operations on a matrix do not change the lin dep/indep of the rows.

Example 1. Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Also suppose col3 = col1 + 2(col2).

Do row op: row2 := row2 + row1. Call the new mtx A'.

Q: Does the relation col3 = col1 + 2(col2) also hold for A'?

Another point of view: rewrite the eqn col3 = col1 + 2(col2) for A as: col1 + 2(col2) - col3 = $\vec{0}$.

Then $A \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \vec{0}$. Why?

Q: Why should $A' \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \vec{0}$? Ans: b/c row ops don't change sols.

More review: def of rank; def of pivot col.

Q: What does "pivot cols of A" mean? Ans: it means those columns that, *after* Gauss-Jordan elimination, *become* pivot cols in rref(A).

Theorem 1. For any $m \times n$ mtx A the pivot cols form a basis for CS(A). So dim(CS(A)) = rank(A).

Example 2. Let $A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 3 & 12 & 4 & 2 \end{bmatrix}$. Do: row3 :=row3 - 3(row1); then, row1 :=row1 - row2; we get:
$\mathrm{rref}(A) = \begin{bmatrix} 1 & 4 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Q: Which cols are free? Is each free col of rref(A) a lin comb of the pivot cols of rref(A)? Yes. This is not a coincidence! Can you see why?

Q: Is each free col of A a lin comb of the pivot cols of A? Yes. Why? By the above theorem, row operations on a matrix do not change the lin dep/indep of the cols. So the lin dep relations that hold for the cols of rref(A) also hold for the cols of A.

Q: Are the pivot cols of A enough to span CS(A)? Yes. Why? b/c the free cols are lin combs of the pivot cols, so they do not "add anything" to the span of the columns.

Q: Are the pivot cols of rref(A) lin indep? Are the pivot cols of A lin indep? Yes. Why? Briefly, b/c they have 1's in different positions. Give rigorous proof...

Q: Are the pivot cols of A lin indep?

Q: Find a basis for CS(A). Ans: First and third cols.

Q: dim(CS(A)) =? Ans: 2.

Summary: Every free col is a lin comb of the pivot cols. Therefore the pivots cols are enough to span CS(A). Also, the pivot cols are lin indep. So they form a basis for CS(A).

Q: How do you find a basis for the row space? Hint: Think about A^T. Ans: RS(A) = CS(A^T). So just find a basis for CS(A^T).

Recall:

Theorem For any mtx A, $\text{rank}(A) = \text{rank}(A^T)$.

Theorem 2. For any matrix A, $\dim(\text{CS}(A)) = \dim(\text{RS}(A))$. In other words, for *any* mtx A, # of lin indep cols = # of lin indep rows!

Proof. $\dim(\text{RS}(A)) = \dim(\text{CS}(A^T)) = \text{rank}(A^T) = \text{rank}(A) = \dim(\text{CS}(A))$. □

Recall: how do we find NS(A)? Ans: take all lin combs of the special sols of $A\vec{x} = \vec{0}$.

So: NS(A) = span(the special sols).

Q: Are the special sols a basis for NS(A)? Yes. Why? They span, and are lin indep. Why are they lin indep? Briefly, they have 1's in different places.

Example 3. Let's continue with the same A as in the above example...

Q: Find the free vars and the special solutions.

Q: Are the special sols lin indep? Ans: Yes, b/c they have 1's in different positions.

Q: Find a basis for $N(A)$. Ans: The special sols. (And this is why they're called *special* sols!)

Theorem 3. For any $m \times n$ mtx A, the special sols to $A\vec{x} = \vec{0}$ form a basis for NS(A). Therefore $\dim(\text{NS}(A)) = \text{nullity}(A)$.

(Proof omitted.)

Proof of Part I of FTLA

Q: For any $m \times n$ mtx A, $\text{rank}(A) + \text{nullity}(A)$ =? Ans: n. So $\dim(\text{RS}(A)) + \dim(\text{NS}(A))$ =? Ans: n.

Q: Prove that LNS(A) = NS(A^T). Proof: LNS(A) = all row vectors $\vec{y}$ s.t. $\vec{y}A = \vec{0}$. Take transpose of both sides: $A^T\vec{y}^T = \vec{0}^T$. $\vec{y}^T$ is a col vector; call it $\vec{x}$. So $\vec{y}$ is in LNS(A) iff $\vec{x}$ is in NS(A^T).

Use this to prove $\dim(\text{CS}(A)) + \dim(\text{LNS}(A)) = m$. (Hint: consider dims of row space and null space of A^T.)

HW # 20, due Wed 7 Nov
Read sec 3.6. Preview 1.2 and 4.1.
Do: p. 161: 1,2,7,8,13bc,15,16.

Always prove or explain all your answers, even if the book doesn't ask for it!