Recall defs: rref; free col, pivot col, rank, nullity.

Today we will study the general case of $Ax = b$, i.e., when $\vec{b} \neq 0$.

Definition 1. In $A\vec{x} = \vec{b}$, if $\vec{b} = \vec{0}$, we say the system is **homogeneous**. Otherwise we say it is **nonhomogeneous**.

Example 1. (a) Find all solutions to the $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Q: Can you find a few vectors such that all the above solutions are lin combs of these few vectors? Yes: the special solutions.

(b) Find all solutions to the equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Q: Can you find a few vectors such that all the above solutions are lin combs of these few vectors? No!

Q: Find a particular sol $\vec{x_P}$ to the nonhomogeneous system.

Q: T or F: If $\vec{x_N}$ is a solution to the homogeneous system, then $\vec{x_P} + \vec{x_N}$ is guaranteed to be a solution to the nonhomog sys. Ans: T. Why?

Q: T or F: *Every* sol to the nonhomogeneous system can be written as $\vec{x_P} + \vec{x_N}$ for some $\vec{x_N} \in \text{NS}(A)$. Ans: T. Why?

This works for any nonhomog system: We can obtain all sols to a non-homog system by knowing just one *particular* sol to it, plus the special sols to the homog system!

general sol = particular sol + lin combs of special sols

Q: Recall For every $m \times n$ mtx, rank $\leq m$ and $\leq n$. Why?

Definition 2. An $m \times n$ mtx A is said to have **full column rank** iff its rank equals its number of columns (r=n); i.e., there is a pivot in every col.
A is said to have **full row rank** iff its rank equals its number of rows (r=m); i.e., there is a pivot in every row.

Q: T or F: Every homog sys has at least one solution. Ans: T; the trivial sol: $\vec{0}$.

Q: T or F: If a homog sys has a nonzero solution, then it has infinitely many solutions. Ans: T. Why? Any multiple of the nonzero sol is also a sol.

Conclusion: Every homog system has either one sol or infinitely many.

Q: Circle all possibilities: A nonhomog system that has more unknowns than eqns can have (1) no sols; (2) only one sol; (3) infinitely many sols. Why?

Ans: no sols, or infinitely many (impossible to have only one). Reason: $n > m$, so there must be at least one free var. So the homog sys has infinitely many sols. Why? Now, we have two cases: The nonhomog sys $A\vec{x} = \vec{b}$ either has sols or doesn't have sols.

Case 1: $A\vec{x} = \vec{b}$ has a sol. Then, the homog sys has infinitely many sols (why?); so $A\vec{x} = \vec{b}$ also has infinitely many sols; why?

Case 2: $A\vec{x} = \vec{b}$ has no sols. Then nothing to prove.

Q: Circle all possibilities: A nonhomog system that has as many unknowns as eqns can have (1) no sols; (2) only one sol; (3) infinitely many sols. Why?

Ans: All three can happen. Reason:

Case 1: there are no free vars. Then rank = ? n; every col is a pivot col. So there is exactly one sol.

Case 2: there are free vars. So the homog sys has infinitely many sols. So like above, either no sols, or infinitely many.

Give examples...

Q: Circle all possibilities: A nonhomog system that has more eqns than unknowns can have (1) no sols; (2) only one sol; (3) infinitely many sols. Why?

Ans: All three can happen. Give examples.

HW # 16, due Mon 29 Oct
Read sec 3.4. Preview sec 3.5.
Do: p. 136: 1, 3, 9, 10abd, 13, 27, 34, 36.

Always prove or explain all your answers, even if the book doesn't ask for it!