

Recall from last time:

After doing elimination (before back-substitution or back-elimination) the matrix is in *echelon* form (and upper-triangular form too? Yes).

*Definition 1.* A matrix is in **echelon form** iff:

1. under every pivot the entries are all zeros; and
2. the pivots are in a “staircase” arrangement, i.e., if  $p$  is a pivot, then every pivot to the right of  $p$  is below  $p$  (more precisely, for any two pivots  $A_{i,j}$  and  $A_{k,l}$ ,  $i < k$  iff  $j < l$ ); and
3. all rows of zeros appear at the bottom, i.e., below all pivots.

After Gaussian Elimination, if we continue with back-elimination, (i.e., we do Gauss-Jordan Elimination) we obtain a matrix in the following form.

*Definition 2.* A matrix is in **reduced row echelon form (rref)** iff:

1. it is in echelon form; and
2. the entries above each pivot are zero; and
3. all pivots equal 1.

The rref of a matrix  $A$  is denoted by  $\text{rref}(A)$ .

*Note.* Our book sometimes just says “reduced form”, instead of “reduced row echelon form.”

Examples. Write various mtxs of various sizes, both in and not in rref.

### Pivot Cols, Free Cols, Rank, Nullity

*Example 1.* Consider the following two systems of equations:

$$\text{System 1: } \begin{cases} x + 2y - z = 3 \\ -2y + z = 1 \end{cases} \quad \text{System 2: } \begin{cases} x + 2y - z = 3 \end{cases}$$

Q: Find all sols for each system. How many sols does sys 1 have? How about sys 2? Ans: each sys has infinitely many sols.

But sys 2 has more sols. We have two free variables in sys 2, and only one free variable in sys 1.

The set of all sols to sys 2 is a 1 dimensional space.

The set of all sols to sys 1 is a 2 dimensional space.

We’ll see what dimension means exactly next week. Today, we’ll see names for the number of free vars, number of pivots, etc.

*Definition 3.* For a mtx in rref, the columns containing the pivots are called the **pivot columns**; the rest of the columns are called the **free columns**.

*Definition 4.* The **rank** of a mtx  $A$  is the number of pivot columns in  $\text{rref}(A)$ . The **nullity** of a mtx  $A$  is the number of free cols in  $\text{rref}(A)$ .

(The word nullity is not in our book.)

Examples:

Q: What are the rank and nullity of  $I_3$ ? How about  $I_n$ ? Ans: 3,0; n,0.

Q: Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . What is  $\text{rref}(A)$ ? Ans: Identity matrix. Why? So what are the rank and nullity of  $A$ ? Ans: 2 and 0.

Q: Is the rank of every  $n \times n$  mtx  $= n$ ? No. Why?

Q: Is the rank of every invertible  $n \times n$  mtx  $= n$ ? Yes. Why?

Q: True or false: for every  $n \times n$  mtx, rank + nullity  $= n$ . Ans: T. Why?

Q: For every  $m \times n$  mtx, rank + nullity  $= ?$ . Ans:  $n$ . Why?

Q: Which gives us the number of the special sols for a mtx, rank or nullity? Ans: nullity.

Q: What is the maximum possible rank of an  $m \times n$  mtx? Ans: the smaller of  $m$  and  $n$ . Why? b/c can have at most one pivot in each row, and in each col.

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Recall: column space and nullspace are both vector spaces. Next week we'll learn the notion of dimension of a vec space. It turns out that

rank = dim of col space

nullity = dim of nullspace.

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*Theorem 1.* For any  $m \times n$  mtx  $A$ ,  $\text{rank}(A) = \text{rank}(A^T)$ .

Proof not easy. Skip.

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Q: T or F: For any  $m \times n$  mtx  $A$ ,  $\text{nullity}(A) = \text{nullity}(A^T)$ ? False! Why?

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**HW # 15, due Fri 26 Oct**

Read sec 3.3. Preview sec 3.4.

p. 128: 1,3,8,9,12(a). p. 120: 15,22,23,27,28,29.

Always prove or explain all your answers, even if the book doesn't ask for it!