Definition 1. Given any mtx A, the set of all vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$ is called the **nullspace** of A, denoted NS(A). (Book uses boldface N.)

Example 1. Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 1 \\ -1 & -3 & 4 \end{bmatrix}$. Find NS($A$).

We need to find all solutions to $A\vec{x} = \vec{0}$. Augment A with a 4th col of zeros. Use Gaussian elimination.

row2=row2-2row1, row3=row3+row1 $\rightarrow \begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 4 & | & 0 \end{bmatrix}$, row3=row3-4row2 $\rightarrow U = \begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

So $z = 0$, and $x + 3y = 0$. Pick y as our free variable. So the set of all solutions is: all vectors of the form $\begin{bmatrix} -3y \\ y \\ 0 \end{bmatrix}$. We call this set of vectors NS(A) .

If we let $y = 1$, then $x = -3$, so we get one solution: $(-3, 1, 0)$. Any other solution will be a multiple of this one. Why?

So we call this solution a *special* solution (even though there isn't really anything special about it!).

So NS(A) = $\{\vec{v} | \ \vec{v} = c(-3, 1, 0)$ for some scalar c$\}$.

Q: Do U and A have the same nullspace? Ans: Yes, because $A\vec{x} = \vec{0}$ and $U\vec{x} = \vec{0}$ have the same solutions. Why?

Q: Which row operations do or do not change the solutions to $A\vec{x} = \vec{b}$? Ans: None of the row ops change the sols. Why? We've seen this before; row operations are "reversible".

Theorem 1. Row operations do not change the nullspace of a mtx. Col operations do not change the column space of a mtx.

Proof: Challenge problem.

Q: Is NS(A) a vector space? For every matrix A? For some but not all? Ans: Yes, for all matrices.

Example 2. Let A be a square mtx with nonzero det. Make up an example. Find NS(A). Ans: The **trivial vector space** $\{\vec{0}\}$ (it contains only the zero vector). Why?

Theorem 2. For any matrix A, NS(A) is a vector space.

Proof. ... ☐

Example 3. Let $A = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 2 & 6 & 1 & -4 \\ -1 & -3 & 4 & 2 \end{bmatrix}$. Find NS($A$).

We need to find all solutions to $A\vec{x} = \vec{0}$. Augment A with a 5th col of zeros.

Do: row2=row2-2row1, row3=row3+row1; we get: $\begin{bmatrix} 1 & 3 & 0 & -2 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 4 & 0 & | & 0 \end{bmatrix}$.

Do: row3=row3-4row2; we get: $\begin{bmatrix} 1 & 3 & 0 & -2 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$.

So $x + 3y + 0z + w = 0$, and $z = 0$. What shall we pick as the free variables? w and y.

So $x = -3y - w$; and $z = 0$. So general solution is of the form $\begin{bmatrix} -3y - w \\ y \\ 0 \\ w \end{bmatrix}$.

So $\mathrm{NS}(A) =$ all vectors of the form $\begin{bmatrix} -3y - w \\ y \\ 0 \\ w \end{bmatrix}$.

Q: Can you find a few specific solutions such that all the above solutions are lin combs of the specific ones you found? Our book calls them **special solutions**. Find them as follows:

Finding the "special solutions": One at a time, let each of the free variables equal 1, and the rest zero.

We picked w and y as free variables. So once let $y = 1$, $w = 0$, and another time do the opposite, let $y = 0$, $w = 1$. For each case, find x and z. These give two "special solutions".

Book writes: **Complete Solution** = y times special sol #1 + w times special sol #2.

Q: Is every vector in $\mathrm{NS}(A)$ a lin comb of these two special sols? Why?

Echelon form

Recall: after doing Gaussian Elimination, the matrix is in what form? Ans: Upper triangular.

Actually, "upper triangular" is not specific or descriptive enough. For example, consider: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Q: Is A upper triangular? Yes.

Q: Is A a matrix one might obtain after doing Gaussian Elimination? No. Why?

Q: Let $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Is B upper triangular? Yes. Is B a matrix one might obtain after doing Gaussian Elimination? No. Why?

Q: Let $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Is B upper triangular? Yes. Is B a matrix one might obtain after doing Gaussian Elimination? Yes.

The shape of matrices one always obtains after Gaussian Elimination is called *echelon form*. Roughly speaking, it means "staircase form". (Echelon comes from old French, meaning rung of a ladder.)

To give a precise def for echelon form, first recall the def of **pivot**: the first non-zero entry in each row.

Definition 2. A matrix is in **echelon form** iff:
1. under every pivot the entries are all zeros; and
2. the pivots are in a "staircase" arrangement, i.e., if p is a pivot, then every pivot to the right of p is below p (more precisely, for any two pivots $A_{i,j}$ and $A_{k,l}$, $i < k$ iff $j < l$); and
3. all rows of zeros appear at the bottom, i.e., below all pivots.

After Gaussian Elimination, if we continue with back-elimination, we obtain:

Definition 3. A matrix is in **reduced row echelon form** (**rref**) iff:

1. it is in echelon form; and
2. the entries above each pivot are zero; and
3. all pivots equal 1.

Note. Our book sometimes just says "reduced form", instead of "reduced row echelon form."

HW # 14, due Fri 26 Oct

Read sec 3.2.

Do: p. 118: 1-4,9,11,12, 16,17,19.

Always prove or explain all your answers, even if the book doesn't ask for it!