Q: Is $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ a linear combination of the columns of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$? Ans: Yes. Why?

Definition 1. The **column space** of an $m \times n$ matrix A is the set (i.e., the collection) of all vectors that are lin combs of the columns of A.

Book writes $\mathbf{C}(A)$ for the col space of A. I'll write CS(A).

Example 1. What is the column space of $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$? Let $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Find CS$(B)$.

Why study col spaces?

Given a mtx A, let's say we want to know what are all possible right-hand sides for which $A\vec{x} = \vec{b}$ has a solution.

• Very important: To ask "for which vectors $\vec{b}$ does $A\vec{x} = \vec{b}$ have a solution" is the same as asking "which vectors $\vec{b}$ are in the col space of A." *Why?*

Q: Suppose the vectors $\vec{b}$ and $\vec{c}$ are in the col space of a mtx A; Which of the following vectors are guaranteed to be in CS(A)? Why?

(i) $\vec{b} + \vec{c}$. (ii) $5\vec{b}$. (iii) $5\vec{b} + 9\vec{c}$. (iv) Any linear combination of $\vec{b}$ and $\vec{c}$.

Theorem 1. The column space of any m by n mtx A satisfies the following two properties:

(i) It is *closed under vector addition*; i.e., for every $\vec{v}$ and $\vec{w}$ in CS(A), $\vec{v} + \vec{w}$ is in CS(A).

(ii) It is *closed under scalar multiplication*; i.e., for every $\vec{v}$ in CS(A) and for any scalar c, $c\vec{v}$ is in CS(A).

Proof: next time.

Q: What is $\mathbb{R}^n$? It is the set of all n-component vectors.

The above two properties are very special and important. We have a special name for when they are satisfied:

Definition 2. Let V be a set of vectors in $\mathbb{R}^n$ (for some n). V is said to be a **vector space** if it satisfies both of the following conditions:

1. It is **closed under vector addition**: for every $\vec{v}$ and $\vec{w}$ in V, $\vec{v} + \vec{w}$ is in V.
2. It is **closed under scalar multiplication**: for every $\vec{v}$ in V and for every scalar c, $c\vec{v}$ is in V.

Note. Sec 3.1 talks about subspaces. This is almost the same thing as a vector space. We'll talk about it next time. For now, just pretend they mean the same thing.

Example 2. Let V be the set of all vectors of the form $\begin{bmatrix} a \\ 2a+1 \end{bmatrix} \in \mathbb{R}^2$. Let W be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix} \in \mathbb{R}^2$. Is V a vector space? How about W? Ans: V no, W yes. Why?

Q: Is the col space of every mtx a vector space? Yes, by Theorem 1 above.

HW #12, due Wed 10 Oct
Read sec 3.1.
Do: p. 108: 10, 12, 13, 19-23, 26, 27, 28.

Always prove or explain all your answers, even if the book doesn't ask for it!