Today: 1. Finish some leftovers from last time. 2. Use determinants to compute areas, volumes. 3. (Optional) Cramer's Rule: use determinants to solve equations, and to find inverses.

Remark. Recall def: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j})$. Our book writes this as $\det(A) = \sum_{j=1}^{n} A_{i,j} C_{i,j}$, where $C_{i,j} = (-1)^{i+j} \det(\hat{A}_{i,j})$.

The coefficient $C_{i,j}$ defined above is called the *cofactor* of the entry $A_{i,j}$. Our book's def of det is the same as the one in class – only the terminology is different: book uses cofactors, we use minors.

Definition 1. Given a matrix A, the **cofactor** of its (i, j)-entry is defined as $C_{i,j} = (-1)^{i+j} \det(\hat{A}_{i,j})$.

Example...

Determinants and Area

Theorem 1. The area of the parallelogram with sides $[a \;\; b]$ and $[c \;\; d]$ is given by the absolute value of the det of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Example 1. Find the area of a triangle whose vertices are given by the points $(1, 1)$, $(2, 3)$, and $(-1, 0)$.

Determinants and Volume

Theorem 2. The volume of a parallelepiped in $\mathbb{R}^3$ whose sides are given by the vectors $[a \;\; b \;\; c]$, $[d \;\; e \;\; f]$ and $[g \;\; h \;\; i]$, is equal to the absolute value of the det of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

parallelepiped = parallel + *epipedon* (face, surface).

Cramer's Rule (Optional)

Suppose we're given a linear system of equations $A\vec{x} = \vec{b}$, where A and $\vec{b}$ are given, and we are to find $\vec{x}$. We have learned how to solve this using Gaussian Elimination. A *longer* way to find $\vec{x}$ is as follows!

Theorem (Cramer's Rule) Given $A\vec{x} = \vec{b}$, the jth coordinate of $\vec{x}$ is given by the formula

$$x_j = \frac{\det(B_j)}{\det(A)}$$

where B_j is obtained by replacing the j-th column of A by $\vec{b}$.

Example 2. Solve the system $\begin{aligned} 2x + 4y &= 1 \\ x + 3y &= 2 \end{aligned}$ using Cramer's Rule.

Ans: ...

Cramer's Rule takes *a lot more* work than Gaussian Elimination to solve a system. So why is it useful? I think because it gives us a *formula* for the solution, as opposed to Elimination, which is only a procedure for finding the solution. (Actually, this doesn't really mean anything! Finding $\det(A)$ and $\det(B_j)$ uses a long procedure anyway. Plus, if $\det(A) \neq 0$, then $\vec{x}$ is given by the formula $A^{-1}\vec{b}$. So I'm not sure what the point of Cramer's Rule is!)

Formula for A^{-1} (Optional)

Theorem If A is an invertible matrix, then A^{-1} is given by

$$(A^{-1})_{i,j} = \frac{C_{j,i}}{\det(A)}$$

where $C_{j,i}$ is the cofactor of $A_{j,i}$.

See book for proof (optional).

HW #11, due Mon 08 Oct
Read p. 234-239. Preview section 3.1
Do p. 241: 17, 19, 20. Ch: 25.

Always prove or explain all your answers, even if the book doesn't ask for it!