Review defs: $A_{i,j}$; commutative, associative, distributive.

Review:
Mtx mult is not commutative.
Mtx mult is associative.
Mtx mult is distributive with respect to addition.
Mtx addition is commutative and associative.

Terminology in the book that you're not expected to know *yet*:
Elimination mtx; elementary mtx; permutation mtx; block mtx; factorization; tridiagonal; determinant (p. 70).

So far in the course we've learned: How to solve systems of eqns using the Gaussian elimination method; and, for dims 3 or less, how to see what's going on pictorially. In higher dimensions, can't see pictures; do algebra only.

Today's outline: 1. Properties of inverse mtxs; 2. How to find inverse mtxs.

Recall: What's the $n \times n$ identity mtx?
Note: There is no $m \times n$ identity mtx (for $m \neq n$).
Notation: I_n, or just I, when dim understood.

Q: What important property does I have?
A: $I\vec{x} = \vec{x}$; $IA = A$; $AI = A$. Give dimensions for each.

Q: if x is an n-component row vec, is it true that $\vec{x}I_n = \vec{x}$? A: Yes.

Recall def of when two mtxs are inverses of each other: ...

Q: In the def, are A and B necessarily square? Yes.

So only square mtxs can have inverses.

Definition 1. A matrix A that has an inverse is called **invertible**, and its inverse is denoted by A^{-1}. A matrix that does not have an inverse is called **non-invertible** or **singular**.
Note. 1. A^{-1} does not mean $1/A$. In fact, dividing by a matrix is meaningless.

2. *Singular* has too different meanings. What are they? (We'll see later that these two definitions are related.)

Example 1. (a) Verify that the following two mtxs are inverses of each other.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3/4 & -1/4 \\ -1/2 & 1/2 \end{bmatrix}$$

(b) Use part (a) to solve the following system: $\begin{aligned} 2x + y &= 1 \\ 2x + 3y &= 0 \end{aligned}$

We do only half of the verification! To verify that A and B are inverses of each other, it is enough to check only one of $AB = I$ or $BA = I$; if one is true, so is the other. (This is not easy to prove. We'll prove it later in the semester.)

Q: What's the inverse of a nonzero 1×1 mtx?

Q: Given invertible mtxs A, B write $(AB)^{-1}$ in terms of A^{-1} and B^{-1}.

Theorem 1. $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. To prove $B^{-1}A^{-1}$ is the inverse of AB, we will show their product is I: $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$. So, by the definition of inverse, $B^{-1}A^{-1}$ is the inverse of AB. □

Gauss-Jordan elimination for finding inverses

Example 2. Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Find $B = \begin{bmatrix} x & u \\ y & v \end{bmatrix}$ s.t. $AB = I_2$.

$AB = I_2$ gives two systems of eqns: $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We could solve each separately, which is ok. But quicker way:
Set up an augmented mtx to solve both simultaneously.

We get: $\begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 2 & 3 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & | & 1/2 & 0 \\ 0 & 2 & | & -1 & 1 \end{bmatrix}$

After it's upper-triangular,instead of back-substitution, we **back-eliminate**: make pivots = 1, then make zeros *above* pivots.

$\begin{bmatrix} 1 & 1/2 & | & 1/2 & 0 \\ 0 & 1 & | & -1/2 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 3/4 & -1/4 \\ 0 & 1 & | & -1/2 & 1/2 \end{bmatrix}$

Lo and behold: the RHS is A^{-1} !

Q: BUT WHY?
A: Solving each system separately would give us x=?, y=?, and u=? , v=?

In the Gauss-Jordan method, we try to put the coeff mtx in **reduced row echelon form** (rref): pivots = 1, and zeros below and above pivots (formal def later). If rref $\neq I$, then the mtx is noninvertible.

Q: What's the inverse of I_2? I_n?

Q: Do all mtxs have inverses? No. Can you think of any? Nonzero mtx?

Example 3. These matrices are noninvertible: $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Formula for inverse of 2x2 mtxs (see Note 4, p. 66)

Theorem 2. If $ad - bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Challenge problem: Derive this formula.

HW #6, due Wed 26 Sep
Read sec 2.5. Skip 2.6. Preview section 2.7.
Do p. 72: 1b,6,7,10,14,18,28bcd; [26]; Ch: 8,11,28a.
Always prove or explain all your answers, even if the book doesn't ask for it!