Review defs: lin comb; equivalent; singular, nonsingular; row operation.

Q: Is this a row operation: row$i$ = row$j + c$(row$k$), where $i \neq j$, and $i \neq k$? A: NO! According to the def, to be a row operation, you need $j = i$.

Q: Let b and c be nonzero scalars, $b \neq 1$. Is this a row operation: row$i = b$(row$i$) + c(row$k$) ? A: NO! But it is OK to do it, since it is a combination of row operations:
1. row$i = b$(row$i$); 2. row$i$ = row$i + c$(row$k$);

Note. Combining several row operations into one is allowed, as long as the result is the same as doing them one by one.

Matrix Addition

Definition 1. Let A be an $m\times n$ matrix (m rows, n columns). If $m = n$, then A is said to be a **square matrix**. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the (i, j)**-entry** of A, denoted by $A_{i,j}$, is the number in the ith row and the jth column of A. We denote the ith row of A by $\mathbf{row}_i(A)$, and the jth column of A by $\mathbf{col}_j(A)$.

Example...

Note. For convenience, some books, including ours, drop the comma from $A_{i,j}$, and instead write A_{ij}. You may do this too, except when it can cause ambiguity, as in: $A_{123} = A_{12,3}$ or $A_{1,23}$?

Q: An m-component column vector is a ?×? matrix? A: $m\times 1$.

Q: An n-component row vector is a ?×? matrix? A: $1\times n$.

Definition 2. Let A and B be $m\times n$ matrices. Then their **sum** $A + B$ is an $m\times n$ matrix C defined by: $C_{i,j} = A_{i,j} + B_{i,j}$.

Example: ...

Dot Product

Q: How do we add vectors to each other? A: Component-wise.

Unfortunately, multiplying vectors is not Component-wise. It even has a special name:

Definition 3. Given two vectors $\vec{v} = (v_1, \cdots, v_n)$ and $\vec{w} = (w_1, \cdots, w_n)$, their **dot product** (also called **inner product**) is defined to be: $\vec{v} \cdot \vec{w} = v_1w_1 + \cdots + v_nw_n$.

Example...

Matrix Multiplication

We add matrices component-wise: $(A+B)_{i,j} = A_{i,j}+B_{i,j}$. But we do not multiply matrices component-wise: $(AB)_{i,j} \neq A_{i,j}B_{i,j}$ (just as vector addition is component-wise, but the dot product isn't).

Definition 4. Let A be an $m\times n$ matrix, and B an $n\times q$ matrix. Then their **product** AB is an $m\times q$ matrix C defined by $C_{i,j} = \text{row}_i(A) \cdot \text{col}_j(B)$. (Equivalently, C can be defined by: $\text{col}_j(C) = A\,\text{col}_j(B)$.)

Examples ...

Q: What type of mtx can be mult by itself? A: A square mtx.

Notation: $AA = A^2, AAA = A^3, \cdots$.

Example 1. Compute BA, where $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Definition 5. The $n \times n$ **identity matrix** I_n is defined to have 1's along its diagonal, and 0's elsewhere.

Definition 6. Two $n \times n$ matrices A and B are said to be **inverses** of each other if $AB = I_n$ and $BA = I_n$.

Example 2. Give two mtxs that are inverses of each other. A: A and B above.

(We'll see more on inverses next time.)

Why are we interested in matrix multiplication?

Used in many branches of mathematics, as well as many many other sciences, e.g.: computer graphics, econ, bio, chem, phys, geo, engineering, etc. We'll see a few applications later (differential eqns, least squares).

Today we'll see one very important application: solving linear systems of eqns.

Example 3. Given the system $\begin{matrix} 2x + 5y = 1 \\ x + 3y = 4 \end{matrix}$, rewrite it as $A\vec{x} = \vec{b}$. Then multiply both sides by the mtx B from example 1 above. What happens?

If instead of 1 and 4, we had say 10 and 25, then we'd get...

Q: So, why are we interested in mtx multiplication?
A: If for a system of eqns we can find the inverse of the coeff mtx, then solving the system is very easy.

Matrix multiplication is associative but not commutative

Q: Suppose A is an $m \times n$ mtx, B $p \times q$. What can you say about m, n, p, q, if both AB and BA are defined? A: $n = p$, $m = q$.

Q: Are AB and BA guaranteed to be the same size? A: No. Why?

Q: Suppose A and B are both $n \times n$ mtxs. Is $AB = BA$ guaranteed to hold? A: No. Proof: Give counterexample.

So we say: Matrix multiplication is usually not **commutative**: $AB \neq BA$.

Theorem 1. Matrix multiplication is **associative**: $(AB)C = A(BC)$ (assuming "sizes match", so that the products are defined).

Proof. Challenge problem: p. 62, #16. □

Theorem 2. Matrix multiplication is **distributive** with respect to addition: $A(B + C) = AB + AC$ (assuming "sizes match", so that the products are defined).

Proof. Challenge problem. □

Q: Is mtx addition associative? (What does this question mean?) A: Yes. $(A + B) + C = A + (B + C)$.

Summary:
Mtx mult is usually not commutative.
Mtx mult is associative.
Mtx mult is distributive with respect to addition.
Mtx addition is commutative and associative.

HW #5, due Wed 19 Sep
Read sec 2.4 (skip block mtxs – will do later). Preview section 2.5
Do p. 59: 1,3,4,6,13,14,19; [5,11,17]; Ch: 16, 24.