A systematic method for solving systems of linear equations

The Method of Elimination. Given: m linear equations in n variables, $x_1, \cdots, x_n$.

Step 1. Solve the 1st equation for one (any) of the variables, say x_1. Use this to eliminate x_1 from the remaining equations (eqs 2-m), by subsitituting for it.

Step 2. Solve the 2nd equation for one (any) of the variables, say x_2. Use this to eliminate x_2 from the remaining equations (eqs 3-m), by subsitituting for it.

Step 3. Repeat this process...

Example 1. Use the method of elimination to solve the following system:

$$x + 2y + 3z = 4$$
$$2x + z = -2$$
$$3x - 3y = 3$$

Step 1. Solve for x; then eliminate x from eqs 2 and 3.
Step 2. Solve for y; then eliminate y from eq 3.
Step 3. Solve for z.
Now "back-substitute" to find y, then x.

An equivalent but quicker method

Example 2. In the above example, instead of Step 1 do the following:

Replace eq2 by [eq2 - 2(eq1)]; and replace eq3 by [eq3 - 3(eq1)].

What happens? x is again eliminated from eqs 2 and 3; we get the same eqs as the ones we obtained after Step 1 above.

Similarly, instead of Step 2 above, what would we do? ...

Example 3. Now we use matrices to save more time.

First write the system in mtx form: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

Recall: The matrix is called the **coefficient matrix**.

We avoid repeatedly writing the variables. Instead, we just keep track of the coeff mtx and the RHS, by putting them into one larger mtx, called the **augmented matrix**: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & -3 & 0 & 3 \end{array}\right]$

We perform the following operations:

eq2 :=eq2 - 2(eq1); or: row2 :=row2 - 2(row1)
row3 :=row3 - 3(row1)
row2 :=(row2)/(-4) (The -4 is called a **pivot**. We'll see more examples...)
row3 :=(row3) + 9(row2)

These give the following results:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & -3 & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -9 & -9 & -9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5/4 & 5/2 \\ 0 & -9 & -9 & -9 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5/4 & 5/2 \\ 0 & 0 & -9+45/4 & -9+45/2 \end{array}\right]$$

Q: What is the new system of eqs? Same solution as before?

Q: What is the new coeff mtx?

Q: What do you notice about the coeff mtx?
A: It's in **upper triangular** form: all the entries below the diagonal are zero.

Q: Now what?

Once the coeff mtx is in upper triangular form, we finish up by **back substitution**:
$z = 6$, ...

The above is called the method of **Gaussian elimination**.

Book uses slightly different but equivalent method (same name). Use whichever you prefer.

Read in book about: pivot; permanent failure, temporary failure.

Definition 1. Suppose we have a system of equations in n variables, $x_1, \cdots, x_n$. An n-component vector $(c_1, \cdots, c_n)$ is said to be a **solution** for the system if substituting c_i for x_i (for all $i = 1, \cdots, n$) simultaneously satisfies all the equations.

Definition 2. A system is called **singular** if it has no solutions or infinitely many solutions. A system is called **nonsingular** if it has exactly one solution.

(Singular kind of means: abnormal, unique, rare, unusual, ...)

Example 4. Solve each of the following linear systems using Gaussian elimination.

(a) $$\begin{aligned} y + z &= 1 \\ 2x + z &= -2 \\ x + 2y + 3z &= 4 \end{aligned}$$

(b) $$\begin{aligned} x - y + z &= 1 \\ 2x - 2y + 5z &= 3 \\ -x + y - z &= -1 \end{aligned}$$

HW #3, due Fri 14 Sep
Read sec 2.2. Preview section 2.3.
Do P. 40: 5,6,7,9,17,19,20,21.