1. (a) Let $S_{x\text{-axis}}$ denote the reflection of $\mathbb{R}^2$ in the x-axis. (We use R for rotations, and hence S for reflections.) Then we can write $S_{x\text{-axis}}(a,b) = (a,-b)$. Make sure you understand this, and then explain it (as if to someone who doesn't).
   (b) Write a formula for $S_{y\text{-axis}}$, the reflection of $\mathbb{R}^2$ in the y-axis.
   (c) Compute the composition $S_{y\text{-axis}} \circ S_{x\text{-axis}}$ by plugging in the formulas from above. Then show that the result is a rotation of the plane around the origin by 180 degrees.
   (d) Do the two reflections $S_{x\text{-axis}}$ and $S_{y\text{-axis}}$ *commute*, i.e., $S_{y\text{-axis}} \circ S_{x\text{-axis}} = S_{x\text{-axis}} \circ S_{y\text{-axis}}$? Prove your answer.

2. Let $S_{x=1}$ and $S_{y=2}$ denote, respectively, the reflections of $\mathbb{R}^2$ in the vertical line $x = 1$ and the horizontal line $y = 2$.

   (a) Write a formula for $S_{x=1}$ and $S_{y=2}$
   (b) Compute the transformation $S_{x=1} \circ S_{y=2}$, and describe what type of translation it is: Is it a reflection? If so, in what line. Is it a rotation or translation? If so, by how much and in what direction? Or is it something else?
   (c) Do $S_{x=1}$ and $S_{y=2}$ commute? Prove your answer.

3. Given two arbitrary lines l and m in the plane, it is cumbersome and tedious to write formulas for the reflections S_l and S_m in the lines and then compute their compositions. So, instead, in this exercise we will try to understand the composition of two reflections by drawing pictures.

   (a) First, recall that Proposition 3.1.5 says: any rigid motion is determined by where it takes any three non-collinear points. Before continuing, read the proposition in the book and make sure you understand how the above statement is really the same as parts (a) and (b) of the proposition!
   (b) Draw two "typical" non-parallel lines l and m in $\mathbb{R}^2$ (by "typical" we mean: not horizontal, not vertical, and not perpendicular to each other). Then pick three non-collinear points P_1, P_2, and P_3, and, as accurately as you can, draw $S_l(P_i)$, $S_m(P_i)$, $S_l \circ S_m(P_i)$, and $S_m \circ S_l(P_i)$, for each $i = 1, 2, 3$. Suggestion: pick three "convenient" non-collinear points: one on l, one on m, and one on $l \cap m$. Then describe what type of translation each of $S_l \circ S_m$ and $S_m \circ S_l$ is.
   (c) Repeat part (b) above with two parallel lines.
   (d) Repeat parts (b) and (c) with the non-Euclid Applet. Discuss how the results are the same or different.

It is known that:

Theorem Every rigid motion of $\mathbb{R}^2$ is either the identity, a reflection, a rotation, a translation, or a glide reflection — there are no other types of rigid motions! Furthermore,

- every rotation can be written as the composition of two reflections in a pair of lines intersecting at the point fixed by the rotation; and
- every translation can be written as the composition of two reflections in a pair of parallel lines perpendicular to the direction of the translation.

This theorem is very important, and in a way summarizes the most important parts of Section 3.1. But, unfortunately (or fortunately, depending on your point of view), we will not have time to prove it. So let's try to at least understand it well, through the following exercises:

4. Given distinct points A and B in $\mathbb{R}^2$, let T_{AB} denote the translation that takes A to B.

   (a) If $A = (2, 5)$ and $B = (2, 11)$, write a formula for T_{AB}.

   (b) If $A = (-2, 5)$ and $B = (2, 11)$, write a formula for T_{AB}.

   (c) By the above theorem, for every pair of distinct points A and B in $\mathbb{R}^2$, there exist parallel lines l and m such that $T_{AB} = S_m \circ S_l$. Explain how you would find l and m with a compass and straightedge, given A and B. Then try your "construction" on the non-Euclid Applet to see if it works in Hyperbolic Geometry as well.

5. Given a point O and a real number θ between 0 and 180, let $R_{O,\theta}$ denote the rotation of $\mathbb{R}^2$ around O by θ degrees counterclockwise. By the above theorem, $R_{O,\theta}$ can be obtained as the composition of two reflections. Given O and an angle measuring θ, explain how you would find, with a compass and straightedge, lines l and m such that $R_{O,\theta} = S_m \circ S_l$. Then try your "construction" on the non-Euclid Applet to see if it works in Hyperbolic Geometry as well.

6. Determine whether each of the following statements is true or false. Justify your answer by providing either an appropriate diagram or an informal (but convincing) explanation, or both.

   (a) Every rigid motion other than the identity has at most two fixed points.

   (b) Every rigid motion that has no fixed points is a translation.

   (c) The composition of two reflections in two distinct lines is never the identity.

   (d) If three distinct points are fixed by a rigid motion other than the identity, then they are collinear.

   (e) The composition of two reflections in two distinct lines is either a rotation or a translation.

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7. Recall that in the proof of Theorem 8.3.3 (page 320), the assumption that $AB > DE$ *and* $AC > DF$ is not justified. There are at least three ways to fix this; find at least one.