Definitions

Undefined Terms: **point, line, plane**.

Definition 1. Given points A, B, C on a line L, we say B is **between** A and C, written A-B-C or C-B-A, if $f_L(A) < f_L(B) < f_L(C)$ or $f_L(C) < f_L(B) < f_L(A)$.

Definition 2. The segment from A to B is defined as: $\overline{AB} = \{C \mid A\text{-}C\text{-}B\}$

Definition 3. The **distance** between two points A and B on a line L is defined to be $AB = |f_L(A) - f_L(B)|$.

Definition 4. The **length** of $\overline{AB}$ is AB. Two segments are **congruent** if they have the same length.

Definition 5. The **segment** from A to B is defined as $\overline{AB} = \{A, B\} \cup \{X \mid A\text{-}X\text{-}B\}$.

Definition 6. The **ray** from A to B, denoted $\overrightarrow{AB}$, is the set of all points C on $\overleftrightarrow{AB}$ such that A is not between B and C. A is said to be the **endpoint** of $\overrightarrow{AB}$.

Definition 7. Let A, B, C be three noncollinear points. Then the **angle** formed by A, B, C, with **vertex** B, is $\angle ABC = \angle CBA = \overrightarrow{BA} \cup \overrightarrow{BC}$.

Definition 8. The **interior** or **inside** of $\angle BAC$ is the intersection of H and H', where H is the half-plane determined by $\overleftrightarrow{AB}$ containing C, and H' is the half-plane determined by $\overleftrightarrow{AC}$ containing B.

Definition 9. Two angles are said to be **congruent** if they have the same measure.

Definition 10. Two angles $\angle ABC$ and $\angle CBD$ are said to be **supplementary** if A-B-D.

Definition 11. Two angles $\angle AXB$ and $\angle CXD$ are said to be **vertical** if A-X-C and B-X-D.

Definition 12. Two triangles are **congruent** if there is a correspondence between their vertices such that their corresponding sides and angles are congruent.

Definition 13. Let ABC be a triangle, and suppose B-C-D. Then $\angle ACD$ is called an **exterior angle** for the triangle. Angles $\angle ABC$ and $\angle BAC$ are called **remote interior angles** for $\angle ACD$.

Definition 14. A **right angle** is an angle whose measure is $90°$. Two lines are **perpendicular** if they form right angles with each other. The **perpendicular bisector** of a segment $\overline{AB}$ is the line that passes through the midpoint of $\overline{AB}$ and is perpendicular to $\overleftrightarrow{AB}$.

Definition 15. A set S of points is **convex** if $\forall A, B \in S, \overline{AB} \subset S$.

Definition 16. If a line L intersects two other lines M and N, then L is a **transversal** for M and N.

Definition 17. Let A, B, C, A', B', and C' be points such that A-B-C, A'-B'-C', and A and A' are on the same side of $\overleftrightarrow{BB'}$. Then angles $\angle ABB'$ and $\angle BB'C'$ are **alternate interior angles**. If B-B'-B'', then angles $\angle ABB'$ and $\angle A'B'B''$ are **corresponding angles**.

Definition 18. Two lines L and M are **parallel**, written $L||M$, if they do not intersect. Two segments $\overline{AB}$ and $\overline{CD}$ are **parallel** if $\overleftrightarrow{AB}||\overleftrightarrow{CD}$.

Definition 19. A **trapezoid** is a quadrilateral with at least two parallel sides.

Definition 20. Two polygons are **similar** if there is a correspondence between their vertices such that corresponding angles are congruent and the ratios of corresponding side lengths are equal.

Definition 21. A **Saccheri quadrilateral** is a quadrilateral whose base angles are right angles and whose heights are congruent.

Definition 22. The **sphere** of **radius** r **centered** at c is the set of points in $\mathbb{R}^3$ at distance r from c.

Definition 23. Let a circle C be the intersection of a sphere S with a plane P. If P passes through the center of S, then C is called a **great circle** or a **geodesic** for that sphere. Otherwise C is called a **small circle**.

Definition 24. (Informal) On a globe, the small circles that are parallel to the equator are called **longitudes**. The great (semi-) circles through the North and South poles are called **longitudes**.

Definition 25. Two points A and B on a sphere are called **opposite** points if $\overleftrightarrow{AB}$ passes through the center of the sphere The opposite of a point A is denoted $-A$.

Definition 26. If A and B are non-opposite points on a sphere, then the shorter arc of the great circle passing through A and B is denoted $\widehat{AB}$.

Definition 27. Let A, B, C be three points on a sphere, not all on the same geodesic. Then $\widehat{AB} \cup \widehat{BC} \cup \widehat{CA}$ is a **spherical triangle**.

Definition 28. A **lune** is a subset of a sphere whose boundary consists of two great semi-circles.

Definition 29. (Informal) A map f from a sphere to the Euclidean plane is **perfect** if, after possibly rescaling the plane, if necessary, f is an isometry (see next definition).

Definition 30. An **isometry** is a map f (between two metric spaces) that *preserves distance*, i.e., $\forall A, B$, $AB = f(A)f(B)$.

Definition 31. A **glide reflection** is the composition of a translation parallel to a line L followed by a reflection in L.

Definition 32. Two isometries S and S' are **inverses** of each other if $S' \circ S = S \circ S' = I$. The inverse of S is denoted S^{-1}.

Definition 33. Let $C \in \mathbb{R}^2$ and $r > 0$. The **central similarity** $S_{C,r}$ is defined by: $\forall P \in \mathbb{R}^2$, $S_{C,r}(P) = P'$, where $P' \in \overrightarrow{CP}$ and $CP' = rCP$. C is called the **center** of the transformation, and r the **ratio of similarity**.

Definition 34. A **similarity** transformation R is the composition of a central similarity S with an isometry T, i.e., $R = S \circ T$.

Axioms

Axiom 1. A line is a set of points. Any two distinct points are contained in exactly one line.

Axiom 2. (Ruler Axiom) For every line L there exists a bijection $f_L : L \to \mathbb{R}$.

Axiom 3. (Pasch's separation axiom for lines; includes definition of *half-plane*) Any line L divides the plane (i.e., the plane $- L$ can be partitioned into) two sets, called **half-planes** or **sides**, such that: (a) for any two points A, B in the same half-plane, $\overline{AB}$ lies in that half-plane, and (b) for any two points A, B in different half-planes, $\overline{AB}$ intersects L.

Axiom 4. (Angle measurement; includes definition of *measure*) To every angle $\angle ABC$ there corresponds a real number strictly between 0 and 180, called its **measure** (in degrees), denoted $\text{m}\angle ABC$, or $\measuredangle ABC$.

Axiom 5. (Angle construction) Let H denote one side of a line $\overleftrightarrow{AB}$. Then for every $r \in (0, \infty)$ there is exactly one ray $\overrightarrow{AC}$ such that $C \in H$ and $\text{m}\angle BAC = r$.

Axiom 6. (Angle addition) If D is a point in the interior of an angle $\angle ABC$, then $\text{m}\angle ABC = \text{m}\angle ABD + \text{m}\angle DBC$.

Axiom 7. (Supplementary angles) Measures of supplementary angles add up to 180 degrees.

Axiom 8. (SAS) If two sides and the angle between them in one triangle are congruent to two sides and the angle between them in another triangle, then the two triangles are congruent.

Axiom 9. (Euclid's parallel axiom, A.K.A. Euclid's 5th postulate) Given any line L and any point $p \notin L$, there is at most one line that contains p and is parallel to L.

Axiom 15. To every polygon there corresponds a positive real number called its **area**.

Axiom 16. Congruent triangles have equal areas.

Axiom 17. Suppose R and R' are polygonal regions (i.e., polygons with their interiors) such that $R \cap R'$ contains no interior points of R or R'. Then area of $R \cup R'$ equals area of R plus area of R'.

Axiom 18. The area of a rectangle $ABCD$ equals $(AB)(CD)$.

Hyperbolic Parallel Axiom If $P \notin L$, then there is more than one line through P parallel to L.

Theorems

Theorem 1.1 Two lines intersect in at most one point.

Theorem 1.2 1. AB=BA. 2. If A-B-C, then $AB+BC = AC$. 3. For any three distinct points on a line, exactly one of them is between the other two.

Theorem 1.3 (Extendibility) Any segment $\overline{AB}$ can be extended on either side by any given positive distance.

Theorem 1.4 (Midpoint Theorem) Every segment $\overline{AB}$ has a **midpoint** $M \in \overline{AB}$, i.e., $AM = MB$.

Theorem 1.5 $\overrightarrow{AB} = \overline{AB} \cup \{X|\ A\text{-}B\text{-}X\}$.

Theorem 1.6 Let L be a line. If $A \in L$ and $B \notin L$, then $\overrightarrow{AB} - \{A\}$ lies on the same side of L as B.

Theorem 1.7 If A, B, C are not collinear and A-C-D and B-M-C and A-M-E, then E is in the interior of $\angle BCD$.

Theorem 1.8 Vertical angles are congruent.

Theorem 2.1 If, in a triangle ABC, $AB = AC$, then $\text{m}\angle ABC = \text{m}\angle ACB$.

Theorem 2.2 (Exterior angle inequality) An exterior angle of a triangle is greater than any of its remote interior angles.

Theorem 2.3 (AAS) If two angles and a side not between them in one triangle are congruent to two angles and a side not between them in another triangle, then the two triangles are congruent.

Theorem 2.4 (SSS) If three side of one triangle are congruent to three sides of another triangle, then the two triangles are congruent.

Theorem 2.5 (ASA) If two angles and the side between them in one triangle are congruent to two angles and the side between them in another triangle, then the two triangles are congruent.

Theorem 2.6 A point C is equidistant from points A and B iff C lies on the perpendicular bisector of $\overline{AB}$.

Theorem 2.6.5 (The Triangle Inequality) For any triangle ABC, $AB + BC > AC$.

Theorem 2.7 If P and Q are on the same side of line L, then the shortest two-segment path from P to Q via L is one where the angle of incidence has the same measure as the angle of reflection.

Theorem 2.8 The perpendicular bisectors of the three sides of a triangle meet at a single point.

Theorem 2.9 Let L, M, N be lines such that M is perpendicular to L and N, and N intersects M at a point $P \notin L$. Then N is parallel to L.

Theorem 2.10 Two lines cut by a transversal are parallel iff the alternate interior angles are congruent.

Theorem 2.11 In every parallelogram opposite sides are congruent and opposite angles are congruent.

Theorem 2.12 A non-self-intersecting quadrilateral whose opposite sides are congruent is a parallelogram.

Theorem 2.13 The angle-sum of every triangle is 180°.

Theorem 2.14 If P, Q, R are points on a circle, then $\measuredangle PQR = 90°$ iff $\overline{PR}$ is a diameter of the circle.

Theorem 2.15 The area of a parallelogram $ABCD$ is given by $(AB)h$, where h is the distance between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Theorem 2.16 The area of a triangle is 1/2 times base times height. More precisely, the area of a triangle ABC is given by $(1/2)(AB)h$, where h is the distance between $\overleftrightarrow{AB}$ and the point C.

Theorem 2.17 The area of a trapezoid is $(1/2)(b+b')h$, where b and b' denote the lengths of two parallel sides, and h the distance between them.

Theorem 2.18 The area of a triangle ABC in terms of the (x, y) coordinates of its vertices is $(1/2)|A_xB_y + B_xC_y + C_xA_y - C_xB_y - B_xA_y - A_xC_y|$.

Theorem 2.19 and 2.20 Suppose in a triangle ABC, A-M-B and A-N-C. Then $\overleftrightarrow{MN}||\overleftrightarrow{BC}$ iff $AM/AB = AN/AC$.

Theorem 2.21 (AAA for similarity) If corresponding angles in two triangles are congruent, then the two triangles are similar.

Theorem 2.22 (SAS for similarity) If ABC and PQR are triangles such that $AB/PQ = BC/QR$ and $\measuredangle B = \measuredangle Q$, then ABC and PQR are similar.

Theorem 2.23 (Pythagoras) If in a triangle ABC $\measuredangle B = 90°$, then $AB^2 + BC^2 = AC^2$.

Theorem 3.1 The sum of any two angles of a triangle is less than two right angles.

Theorem 3.2 (Saccheri-Legendre) Given any triangle ABC, there is another triangle such that one of its angles has measure $(1/2)\measuredangle A$.

Theorem 3.3 The angle sum of any triangle is $\leq 180°$.

Theorem 3.4 (Strong exterior angle inequality) The measure of every exterior angle of a triangle is greater than or equal to the sum of the measures of its two remote interior angles.

Theorem 3.5 In hyperbolic geometry, there is a triangle with angle sum $< 180°$.

Theorem 3.6 In hyperbolic geometry, every triangle angle sum $< 180°$.

Theorem 3.7 If M_{AB} is the shortest path between points A and B on a sphere S, there is a plane P passing through the center of S such that $M_{AB} \subset P \cap S$.

Theorem 3.8 Every shortest path on a sphere is an arc of a great circle.

Theorem 3.9 Any two geodesics have exactly two points in common.

Theorem 3.10 If ABC is a spherical triangle, then $(-A)(-B)(-C)$ is a congruent spherical triangle, with the same area.

Theorem 3.11 A sphere of radius R has area $4\pi R^2$.

Theorem 3.12 A lune with corner angle θ has area $2\theta R^2$.

Theorem 3.13 (Excess Theorem) For every spherical triangle ABC, $\measuredangle A + \measuredangle B + \measuredangle C = \pi + \text{area}(ABC)/R^2$.

Theorem 3.14 There is no isometry from a sphere to the Euclidean plane.

Theorem 4.1 Every reflection is an isometry.

Theorem 4.2 and 4.3 If f is an isometry, then A-B-C iff $f(A)$-$f(B)$-$f(C)$.

Theorem 4.4 If f is an isometry, then every triangle is congruent to its image under f.

Theorem 4.5 Angles are invariant under isometry.

Theorem 4.6 Parallelism is invariant under isometry.

Theorem 4.7 The composition of two isometries is an isometry.

Theorem 4.8 If triangle ABC is congruent to DEF, then there is an isometry that takes A to D, B to E, and C to F.

Theorem 4.9 The composition of two translations is a translation.

Theorem 4.10 If M_L and $M_{L'}$ are reflections in two parallel lines L and L' a distance d apart, then $M_{L'} \circ M_L$ is a translation by $2d$ along a line perpendicular to L, in the direction from L towards L'.

Theorem 4.11 Let M_L and $M_{L'}$ be reflections in two lines L and L' that intersect in a point P. Let θ be the angle formed in the clockwise direction from L to L'. Then $M_{L'} \circ M_L = R_P(2\theta)$.

Theorem 4.12 Composition of isometries is associative.

Theorem 4.12.5 (a) (Example 4.10) The composition of a translation along a line L followed by a reflection in a line that's not perpendicular to L is a glide reflection. (b) The composition of a translation along a line L followed by a reflection in a line that's perpendicular to L is a reflection.

Theorem 4.13 Any three circles with noncolinear centers intersect at most once.

Theorem 4.14, 4.15 Every isometry is determined by where it maps any three noncolinear points. Every isometry that fixes three noncolinear points is the identity.

Theorem 4.16 Every isometry is the composition of at most three reflections.

Theorem 4.17 An isometry that fixes at least two points A and B must either be the identity or the reflection in the line $\overleftrightarrow{AB}$.

Theorem 4.18 An isometry that fixes exactly one point A is the composition of two reflections in two lines passing through A.

Theorem 4.19 An isometry with no fixed points is the composition of at most three reflections.

Theorem 4.20 (Classification of Isometries) Every isometry is the identity, a reflection, a translation, a rotation, or a glide reflection.

Theorem 4.21 The composition of two isometries reverses sense iff exactly one of them reverses sense.

Theorem 7.1 Let P' and Q' be images of points P and Q, respectively, under a central similarity $S_{C,r}$. Then $P'Q' = rPQ$, and $\overleftrightarrow{P'Q'} || \overleftrightarrow{PQ}$.

Theorem 7.2 The images of three collinear points under a central similarity are collinear.

Theorem 7.3 The image of a circle under a central similarity is a circle.

Theorem 7.4 The image of a triangle under a similarity is a triangle similar to the original triangle.

Theorem 7.5 If a triangle ABC is similar to a triangle PQR, then there is a similarity S such that $S(A) = P$, $S(B) = Q$, and $S(C) = R$.

Classification of the theorems:

E=Euclidean, H=Hyperbolic, N=Neutral

1.1-2.9:N ; 2.10:if:N,only-if:E ; 2.11:E ; 2.12:N ; 2.13-2.20:E ; 2.21:N ; 2.22-2.23:E ; 3.1-3.4:N ; 3.5-3.6:H ; 3.7-3.8:E(+H) ;