Final Exam. Math 450, Models of Computation. Name: _____
Instructor: Ramin Naimi Tue 8 Apr 2001

Do **only six** of the following problems. 20 points per problem. Closed book. Closed Notes. Please write very legibly.

In the following problems, unless stated otherwise, you may use Church's Thesis freely. Also, except for the first problem, you may freely use the fact that '$x \in W_x$' is not decidable.

1. Use a diagonal argument to show the predicate '$x \in W_x$' is not decidable.
2. Is the predicate '$x \notin W_x$' partially decidable? Prove your answer.
3. (a) Suppose $M(x)$ is a partially decidable predicate. Show if the predicate 'not $M(x)$' is also partially decidable, then $M(x)$ is decidable.
   (b) Find a partially decidable predicate $M(x)$ that's not decidable.
4. Show if f is a one-to-one computable function of one variable, then f^{-1} is computable. (Note: f may be a partial function.)
5. Is there an algorithm that, given any computer program P, decides whether or not 0 is an output of P when given 0 as input? Prove your answer.
6. Is there an algorithm that, given any computable function $f(x)$, decides whether or not range(f) is infinite? Prove your answer.
7. Show that there is a total computable function $f(x)$ such that for every $x \in \mathbb{N}$, $W_x = E_{f(x)}$.
8. State the halting problem, and show it is not decidable.
9. Recall: to say a set $A \subset \mathbb{N}$ is r.e. means the function $f(x) = \begin{cases} 1 & \text{if } x \in A \\ \text{undefined} & \text{otherwise} \end{cases}$ is computable. Prove that the following are equivalent:
   (a) A is r.e.
   (b) The predicate $M(x) =$ '$x \in A$' is partially decidable.
   (c) There is a total bijective computable function $g : \mathbb{N} \to A$.
10. Show that the predicate 'ϕ_x is not total' is not partially decidable.
11. Show that the predicate 'ϕ_x is total' is not partially decidable. Hint: Consider the function
$$g_y(x) = \begin{cases} 1 & \text{if } \phi_y, \text{ when given } y \text{ as input, does not stop in } x \text{ or fewer steps} \\ \text{undefined} & \text{otherwise} \end{cases}.$$
12. (a) Give an algorithm for enumerating all URM programs.
    (b) According to your algorithm, what is the 25th URM program? You don't have to list all of the first 25 programs; just explain how your found the 25th one.
13. True or false? There is a URM program that, given input n in the first register and 0 in all the other registers, stops with the following output: 1 in the nth register, and 0 in all the other registers. Prove your answer.