There are only countably many programs

Lexicographical Ordering

Let's agree on the following ordering of the following seventeen symbols:

J S T Z 0 1 2 3 4 5 6 7 8 9 , ( )

Given any number of strings of the above symbols, we can order them lexicographically (i.e., in "dictionary order").

Example
Order the following strings lexicographically.
JS3T67)0
ZZ,,
1234
(((
ZZ,,,
JS4T

Similarly, given any number of URM programs, P_1, P_2, $\cdots$, P_n, we can order them lexicographically, by treating each program as just a string of symbols.

Example Order the following five programs.
P_1 = J(1,2,1) S(1)
P_2 = Z(1) S(1) S(1)
P_3 = J(1,1,2)
P_4 = T(1,2) T(1,2)
P_5 = Z(1)

Now, suppose we want to systematically list *all* possible URM programs. How would you do this?
Ans: For each $n = 1, 2, \cdots$, list all the n-symbol programs lexicographically. (So the first program would be S(1), which is a 3-symbol program.)
Let's try and write the first ten or twenty.
...

Q: Do you think *you* could, with enough time and patience, find the 100th URM in this ordering? How about the 1000th program?

Q: Do you think *you* could find "the number" for any given URM program ("the number" according to this ordering)?

Q: Is the set of all URM programs countable or uncountable?

Theorem
The set of all computable functions from **N** to **N** is countable.

Proof: Every computable function from **N** to **N** is computed by at least one (and in fact more than one) URM program. From above, we know that there are only countably many URM programs. So, ...

<u>Theorem</u>

There exist functions from **N** to **N** which are not computable.

Proof: If every function from **N** to **N** were computable, then by the above theorem, the set of all functions from **N** to **N** would be countable. But there are uncountably many functions from **N** to **N** (HW). So there exist non-computable functions.

In fact, there exist infinitely many non-computable functions (why?). And worse than that, there exist uncountably many non-computable functions (why?), while there are only countably many computable functions. So in an imprecise sense, most functions are not computable. But fortunately, most functions that come up in "everyday life" are computable!

HW #8, due Monday 12 Mar

Read p. 79-80. **Do:** p. 80: 1.

And the following problems:

Q1: Prove there are uncountably many functions from **N** to **N**.

Q2: The English language has finitely many words (Webster's Dictionary says about 2 million!)

(a) Let T be the set of all three-word English sentences (e.g.: I love chocolate). Is T finite or infinite?

(b) Let S be the set of all 1000-word English essays. (I don't think I'll give an example of that one here!) Is S finite or infinite (i.e., are there only finitely many different 1000-word essays that can ever be written, or infinitely many)?

(c) The set of all finite (but arbitrary length) essays is obviously not finite. Is it denumerable or uncountable? Prove your answer (informally, but carefully and convincingly).