Lab 3: Limits, Continuity, and Differentiability

Objectives:

1. To become familiar with the program **Derive**.
2. To examine the concepts of limit, and continuity.

Note: There is no report to turn in for this lab. You should, however, make sure to learn well how to use the program *Derive* — you will be tested on it individually (not in teams!) later.

Introduction: Using *Derive*

The lab this week will introduce you to another computing resource which we will use in this course. *Derive* is avaliable on the network under the *Mathematics* icon. It specializes in symbolic computation. "Symbolic computation" refers to computing with symbols as well as with numerical values. For example, *Derive* is able to factor $x^2 - 1$ as $(x+1)(x-1)$. The TI-83 is unable to do this.

Derive also has rather nice features for exploring graphs of functions. This is what we will use it for today. To begin, click on the *Derive* icon under *Mathematics*. A screen will appear with a list of menu options and buttons at the top. This particular screen is called the "Algebra" window in *Derive* because this window will be used to author and modify algebraic and other expressions. The first thing we will be doing is looking at the graph of the function $p(x) = \sin(x)/x$ for values of x near 0.

1. First, without using the computer or the calculator, find the domain of $p(x) = \sin(x)/x$.

To enter an expression in *Derive*, select **Author** in the algebra window, then **Expression**. Type $\sin(x)/x$ in the authoring window, then $<$ *Enter* $>$ it.

To plot the function whose rule is given by an expression, make sure the expression is *highlighted* in the algebra window, then click on the second button from the right in the toolbar. Do this now.

You should now see a "graphics" window. This window will have a pair of axes marked with tickmarks and its own menu at the top. Now select **Plot** from this menu. The graph of this function should appear.

Note that although $p(0)$ is undefined (why?), *Derive* does not seem to show it on the graph.

Exploring a Graph with *Derive*

Derive has several features which allow you to explore graphs. First, notice the cross-hairs. They can be controlled with either the mouse or the "arrow" keys. At the bottom of the screen you will see the x- and y-coordinates changing as you move the cross-hairs around.

Now examine the bottom of the screen more closely. The spacing between the tickmarks on the x and y axes will appear as **Scale** in the format *x-scale*: *y-scale*. What are these values now?

There are several features of the graphics window menu which we will also be using. Select **Set**, then **Center**. Type 0 for the *Horizontal* coordinate and 1 for the *Vertical* coordinate, then $<$ *Enter* $>$. Describe what happens. (Also note the *Center* box at the bottom of the screen.)

The other feature we will be using is **Zoom**. Various sorts of zooming are possible. These are performed by the buttons at the right side of the menu bar with little arrows on them. Find and select the button which *zooms in on both axes.* Describe what happens. Pay particular attention to the values for the x and y scales.

You now know the basics of working with *Derive*. During the rest of the lab, we will be using the following sequence of operations to focus on certain points of the graph of a function:

Move the cross-hairs to the point of interest, then **Center** the window on that point, then **Zoom** in on the center of the window.

Try zooming in and out on various points just to get the hang of this sequence of operations.

Limits

Before continuing with the function $p(x) = \sin(x)/x$, let's first examine more closely the concept of "limits" by computing $\lim_{x \to 0} \sin(x)$. We do this in two steps:
(i) Find the limit as x approaches zero from the **right**: $\lim_{x \to 0^+} \sin(x)$;
(ii) Find the limit as x approaches zero from the **left**: $\lim_{x \to 0^-} \sin(x)$.
Use the table below to perform these two steps.

x	$\sin(x)$	x	$\sin(x)$
1		-1	
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	

(i) $\lim_{x \to 0^+} \sin(x) = ?$

(ii) $\lim_{x \to 0^-} \sin(x) = ?$

What does this say about $\lim_{x \to 0} \sin(x)$?

2. Recall we are working with the function $p(x) = \sin(x)/x$. Use *Derive* and/or your calculator to estimate $\lim_{x \to 0} p(x)$. Look at $p(x)$ for the following sequences of x values approaching $x = 0$ from the left and from the right.

x	$p(x)$	x	$p(x)$
-1		1	
-0.1		0.1	
-0.01		0.01	
-0.001		0.001	

Do you think you would get the same results for other sequences of x values approaching 0 from below or from above? *Zooming in* on the graph with *Derive* may help you decide. Based on your conclusion, determine the following limit or explain why it does not exist.

$$\lim_{x \to 0} p(x) =$$

Continuity

Informally, a function is *continuous* at a point if its graph is unbroken there. Another way to think of this is: a function is continuous if you can draw its graph without lifting the pencil off the paper. This idea can also be expressed in terms of limits.

Definition: A function $g(x)$ is *continuous* at a if $\lim_{x \to a} g(x) = g(a)$.

3. Based on this definition, is $p(x) = \sin(x)/x$ continuous at $x = 0$? *Hint: What is the value of $p(x)$ at $x = 0$?*

4. Complete the following definition so that the function $q(x)$ *is* continuous at $x = 0$:

$$q(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ ? & \text{if } x = 0 \end{cases}$$

Other functions

5. For each of the following functions, find all discontinuities and decide which are removable. Also, find all asymptotes. You should first do everything without the aid of a computer or calculator! Then use *Derive* to check your answers and sketch a graph of the function. Hint: *Derive* (and your calculator) may not be able to graph some of the functions that contain very large numbers, or they may even give you a wrong graph! Try graphing a similar function with smaller numbers, as a guide.

(a) $e^{1/x}$

(b) $\ln|x - 10^{200}|$

(c) $x \ln|x|$

(d) $[\sin(\frac{x}{10^{50}})]^{-1}$

(e) $\dfrac{10^{100} - \sqrt{x + 10^{200}}}{x}$