ALTERNATIVE PROOFS ON THE 2-ADIC ORDER OF STIRLING NUMBERS OF THE SECOND KIND

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Abstract

An interesting 2-adic property of the Stirling numbers of the second kind $S(n,k)$ was conjectured by the author in 1994 and proved by De Wannemacker in 2005: $\nu_2(S(2^n,k)) = d_2(k) - 1$, $1 \leq k \leq 2^n$. It was later generalized to $\nu_2(S(c2^n,k)) = d_2(k) - 1$, $1 \leq k \leq 2^n$, $c \geq 1$ by the author in 2009. Here we provide full and two partial alternative proofs of the generalized version. The proofs are based on non-standard recurrence relations for $S(n,k)$ in the second parameter and congruential identities.

Keywords: Stirling numbers of the second kind; congruences and divisibility; Bernoulli numbers

1. Introduction

The study of $p$-adic properties of Stirling numbers of the second kind offers many challenging problems. Let $k$ and $n$ be positive integers, and let $d_2(k)$ and $\nu_2(k)$ denote the number of ones in the binary representation of $k$ and the highest power of two dividing $k$, respectively. Lengyel [5] proved that

$$\nu_2(S(2^n,k)) = d_2(k) - 1$$

for all sufficiently large $n$ (e.g., $k - 2 \leq n$), and conjectured that $\nu_2(S(2^n,k)) = d_2(k) - 1$, for all $k : 1 \leq k \leq 2^n$ which was proved in

Theorem 1 [3, Theorem 1]. Let $k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$. Then we have

$$\nu_2(S(2^n,k)) = d_2(k) - 1.$$
At the very heart of the proof, there is an appealing recurrence for the Stirling
numbers of the second kind involving a double summation

\[ S(n + m, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k - i)!}{(k - j)!} S(n, k - i) S(m, j). \]  

The generalization of Theorem 1 and De Wannemacker’s proof can be found in [7].

**Theorem 2 ([7]).** Let \( c, k, n \in \mathbb{N} \) and \( 1 \leq k \leq 2^n \), then

\[ \nu_2(S(c2^n, k)) = d_2(k) - 1. \]  

In this paper we use Kummer’s theorem on the \( p \)-adic order of binomial coefficients.

**Theorem 3 (Kummer (1852)).** The power of a prime \( p \) that divides the binomial coefficient \( \binom{n}{k} \) is given by the number of carries when we add \( k \) and \( n - k \) in base \( p \). In another form, \( \nu_p\left(\binom{n}{k}\right) = n - dp(n) - k - dp(k) = \frac{dp(k) + dp(n - k) - dp(n)}{p - 1} \) with \( dp(n) \) being the sum of the digits of \( n \) in its base \( p \) representation. In particular, \( \nu_2\left(\binom{n}{k}\right) = d_2(k) + d_2(n - k) - d_2(n) \) represents the carry count in the addition of \( k \) and \( n - k \) in base 2.

We will also need

**Theorem 4 ([3], Theorem 3).** Let \( k, n \in \mathbb{N} \) and \( 1 \leq k \leq n \). Then

\[ \nu_2(S(n, k)) \geq d_2(k) - d_2(n). \]  

This can be proven by an easy induction proof. Note that in general,

**Theorem 5 ([6]).** For every prime \( p \geq 3 \) and integer \( k : 1 \leq k \leq n - 1 \),

\[ \nu_p(S(n, k)) \geq \frac{dp(k) - dp(n) - (n - k)(p - 2)}{p - 1} + 1. \]  

The main goal of this paper is to suggest alternative methods for proving 2-adic properties of the Stirling numbers of the second kind. In Section 2 we discuss some partial proofs of Theorem 2 while full proofs of Theorems 1 and 2 are presented in Section 3. It is remarkable that both known proofs of Theorems 1 and 2 are based on recurrence relations on \( S(n, k) \) in the second parameter such as (3) and (12) or its generalization (13).
2. Preliminaries and partial answers

In this section we provide alternative partial proofs of Theorem 2 for two sets of values of $k$ that are smaller than the full range $\{1, 2, \ldots, 2^n\}$. The proofs and how the tools, identity (6) and Theorem 8, are used seem to be new.

The two sets are defined by $k \leq n$ and $d_2(k) \leq \nu_2(k)$. Their respective cardinalities are $n$ and the $n + 1$st Fibonacci number $F_{n+1}$. In fact, by counting all values $k$ with a fixed number $s = d_2(k)$ of ones in their binary representations (so that $s \leq \nu_2(k)$), we find that there are $\binom{n}{s}$ such $k$s if $s \geq 2$ and $\binom{n}{1}$ powers of two otherwise. We get that

$$|\{k \mid 1 \leq k \leq 2^n \text{ and } d_2(k) \leq \nu_2(k)\}| = \binom{n}{1} + \binom{n-2}{2} + \binom{n-4}{3} + \binom{n-4}{4} + \cdots = F_{n+1}, \text{if } n \geq 1.$$

Let $\pi(k; p^N)$ denote the minimum period of the sequence of Stirling numbers $\{S(n, k)\}_{n \geq k}$ mod $p^N$. Kwong [4] proved the following

**Theorem 6** ([4]). For $k > \max\{4, p\}$, $\pi(k; p^N) = (p - 1)p^{N + l_p(k) - 2}$, where $p^{l_p(k) - 1} < k \leq p^{l_p(k)}$, i.e., $l_p(k) = \lceil \log_p k \rceil$.

Based on the periodicity property and Euler’s theorem we can obtain

**Theorem 7** ([5], Theorem 2). Let $c$ and $n$ be non-negative integers, with $c$ odd. If $1 \leq k \leq n + 2$ then $\nu_2(k!S(c2^n, k)) = k - 1$, i.e., $\nu_2(S(c2^n, k)) = d_2(k) - 1$.

The latter theorem can be proven in a slightly weakened form by replacing $k \leq n + 2$ with $k \leq n$ as it is shown in the following

**Proof.** By the identity (cf. [8], identity (188) on p. 496),

$$\sum_{d|N} \mu(d)k!S\left(\frac{N}{d}, k\right) \equiv 0 \mod N \tag{6}$$

with any positive integers $k$ and $N$, and $\mu$ denoting the Moebius $\mu$-function. Indeed, we set $N = 2^n, n \geq k$, and get that

$$k!S(2^n, k) - k!S(2^{n-1}, k) \equiv 0 \mod 2^n \tag{7}$$

As above, by periodicity and Euler’s theorem, we know that $\nu_2(k!S(2^n, k)) = k - 1$ for any sufficiently large $n$, and thus, by (7), we immediately have that it holds for any $n \geq k$. This argument easily generalizes for $S(c2^n, k)$ with any $c \geq 1$ odd; although,
there will be $2^{\omega(c)+1}$ terms of the form $\pm k!S(c'2^n, k)$ or $\pm k!S(c'2^{n-1}, k)$ in [7] where $c' \geq 1$ is a divisor of $c$ and $\omega(c)$ denotes the number of different prime factors of $c$. The proof can be completed by an induction on $\omega(c)$. \hfill \Box

Another special case can be treated by the following theorem proved by Chan and Manna [2] in a recent paper.

**Theorem 8** ([2], Theorem 4.2). Let $a, m,$ and $n$ be positive integers with $m \geq 3$ and $n \geq a2^m + 1$. Then

$$S(n, a2^m) \equiv a2^{m-1}\left(\left\lfloor \frac{n-1}{2} \right\rfloor - a2^{m-2} - 1\right) + \frac{1 + (-1)^n}{2} \left(\frac{n}{2} - a2^{m-2} - 1\right) \mod 2^m. \tag{8}$$

This guarantees that we can determine $\nu_2(S(2^n, k))$ for any $k$ with at least as many zeros at the end of its binary representation as the number of ones in it.

**Theorem 9.** Let $k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$ with $\max\{3, d_2(k)\} \leq \nu_2(k)$. Then $\nu_2(S(2^n, k)) = d_2(k) - 1$.

**Proof.** We replace $n$ by $2^n$ in Theorem 8 and write $k$ as $k = a2^m$ with some integer $a > 0$. We assume that $m \geq 3$ and $m \geq d_2(a)$, and $k = a2^m \leq 2^n$, i.e., $n \geq n_0 = \lfloor \log_2(a2^m) \rfloor$. Without loss of generality, we can assume that $a$ is odd and $m = \nu_2(k)$; otherwise, we rewrite $a2^m$ as $a'2^{m'}$ with $a'$ odd and $m' > m \geq d_2(a)$. Both [9] and [10] hold with $a'$ and $m'$ while $n$ and $n_0$ are kept unchanged.

Now we prove that

$$S(2^n, a2^m) \equiv \left(\frac{2^{n-1} - a2^{m-2} - 1}{2^{n-1} - a2^{m-1}}\right) \mod 2^m \tag{9}$$

and

$$\nu_2(S(2^n, a2^m)) = d_2(a) - 1 \tag{10}$$

by applying Theorem 8. Note that $\left\lfloor \frac{2^{n-1}}{2} \right\rfloor - a2^{m-2} - 1$ is even while $\left\lfloor \frac{2^{n-1}}{2} \right\rfloor - a2^{m-1}$ is odd; thus, there is guaranteed at least one carry in the application of Theorem 3 to the binomial coefficient of the first term in (8). This proves (9) which can be further evaluated by the last part of Theorem 3. In fact, we get that

$$\nu_2(S(2^n, a2^m)) = d_2(2^{n-1} - a2^{m-1}) + d_2(a2^{m-2} - 1) - d_2(2^{n-1} - a2^{m-2} - 1)
= (n - n_0 + (l_2(a) - d_2(a) - \nu_2(a) + 1)) + (d_2(a) + \nu_2(a) - 1 + m - 2)
- (n - n_0 - 1 + (m - 2) + 1 + (l_2(a) - d_2(a) + 1))
= d_2(a) - 1 < m \tag{11}$$
with \( l_2(a) = \lceil \log_2(a) \rceil. \)

Note that the above proof does not require any induction (although the proof of Theorem 8 uses induction). In addition, we can generalize the proof to obtain

**Theorem 10.** Let \( c, k, n \in \mathbb{N} \) and \( 1 \leq k \leq 2^n \) with \( \max\{3, d_2(k)\} \leq \nu_2(k) \). Then

\[
\nu_2(S(c2^n, k)) = d_2(k) - 1.
\]

**Proof.** In fact, \( k = a2^m \leq 2^n \) implies that the nonzero binary digits of \( c2^n \) and \( a2^m \) avoid each other (perhaps with the exception of the rightmost one in \( c2^n \) when \( a = 1 \) and \( c \) is odd) and thus, (11) can be easily revised:

\[
\begin{align*}
\nu_2(S(c2^n, a2^m)) &= d_2(c2^{n-1} - a2^{m-1}) + d_2(a2^{m-2} - 1) - d_2(c2^{n-1} - a2^{m-2} - 1) \\
&= (n - n_0 + (l_2(a) - d_2(a) - \nu_2(a) + 1) + d_2(c) + \nu_2(c) - 1) \\
&\quad + (d_2(a) + \nu_2(a) - 1 + m - 2) \\
&\quad - (n - n_0 - 1 + (m - 2) + 1) + (l_2(a) - d_2(a) + 1) + d_2(c) + \nu_2(c) - 1) \\
&= d_2(a) - 1 < m
\end{align*}
\]

\( \square \)

3. **Main result: alternative proofs of Theorems 1 and 2**

We now turn to another approach due to Agoh and Dilcher [1]. They developed an alternative recurrence relation for \( S(n + m, k) \) which relates this quantity to terms involving \( S(n, k')S(m, k - k') \) by means of a single summation rather than a double summation as in (3).

**Theorem 11 ([1]).** For \( r \geq \max\{k_1, k_2\} + 2 \), we have that

\[
\frac{k_1!k_2!(r-1)!}{(k_1 + k_2 + 2)!} S(k_1 + k_2 + 2, r) = \sum_{i=1}^{r-1} (i-1)!(r-i-1)! S(k_1 + 1, i)S(k_2 + 1, r-i). \tag{12}
\]

The paper [1] also contains a generalization of this theorem to \( s \geq 2 \) factors involving Stirling numbers on the right hand side in a summation with \( s - 1 \) summation indices. Theorem [1] is a special case with \( s = 2 \).
We will use the generalization of (12) to $r \geq 1$, cf. [1, identity (6)]. It includes a correction term involving Bernoulli numbers

\[
\frac{(k - 1)!(m - 1)!(r - 1)!}{(k + m - 1)!} S(k + m, r) = \sum_{i=1}^{r-1} (i - 1)!(r - i - 1)!S(k, i)S(m, r - i) \\
+ (r - 1)! \sum_{j=r}^{k+m-1} \left( (-1)^{m} \binom{k - 1}{j - 1} + (-1)^{k} \binom{m - 1}{j - 1} \right) \frac{B_{k+m-j}}{k + m - j} S(j, r)
\]

with $B_n$ being the $n$th Bernoulli number.

Now we present an alternative proof of Theorem [1].

**Proof of Theorem [1]**. We prove by induction on $n$. The base case with $n = 0$ is trivial. We consider the equivalent form $\nu_2(k!S(2^n, k)) = k - 1$ of identity [1]. Let us assume that $\nu_2(k!S(2^t, k)) = k - 1$ for any integers $t$ and $k$ such that $1 \leq t \leq n$ and $1 \leq k \leq 2^t$. We prove the statement for $t = n + 1$. We write $k$ in its binary representation $k = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_{d_2(k)}}$ with $0 \leq b_1 < b_2 < \cdots < b_{d_2(k)}$. We have two cases according whether $k \geq 2^n + 1$ or not.

**Case 1.** First let us assume that

\[
2^n < k \leq 2^{n+1}.
\]

The assumption yields that $b_{d_2(k)} = n$ except for $k = 2^{n+1}$.

We use Theorem [1] with $k_1 = k_2 = 2^n - 1$, $r \geq 2^n + 1$, and switching from the notation $r$ to $k$. After slightly rewriting (12), we obtain

\[
(k - 1)!S(2^{n+1}, k) = \frac{(2^{n+1} - 1)!}{(2^n - 1)!^2} \sum_{i=1}^{k-1} \frac{1}{i(k-i)}i!S(2^n, i) (k-i)!S(2^n, k-i).
\]

With $N = 2^{n+1}$, the first factor on the right hand side of (15) is

\[
\frac{(N - 1)!}{(N/2 - 1)!^2} = \left( \frac{N-1}{N/2} \right) \frac{N}{2}
\]

and there is no carry in the addition of $N/2$ and $N/2 - 1$. This yields an overall 2-adic order of $n$ for the whole expression.

We have two subcases. If $k$ is odd then we note that $i(k-i)$ in the denominator of (15) can decrease the 2-adic order, and the unique largest decrement results from
setting $i$ or $k - i$ to $2^{b_d(k)}$. By the inductive hypothesis, the last four factors at the end of (15) contribute $(i - 1) + (k - i - 1) = k - 2$ to the 2-adic order. Hence, we get that

$$
\nu_2(k(k - 1)!S(2^{n+1}, k)) = \nu_2(k) + n - b_{d_2(k)} + 1 + (k - 2)
= n + k - 1 - b_{d_2(k)} = k - 1. \quad (16)
$$

If $k$ is even and $k \neq 2^{n+1}$ then the factor $i(k - i)$ in the denominator of (15) decreases the 2-adic order the most if we set $i$ or $k - i$ to $2^{b_d(k)}$ which yields that the other factor is an odd multiple of $2^{n(k)}$. No other pair $(i, k - i)$ can reach this decrement. If $i = k/2$ then the corresponding term occurs only once, and the decrement is $2(\nu_2(k) - 1) \leq b_{d_2(k)} + \nu_2(k) - 2$. Thus, the right hand side of (16) changes, and we obtain

$$
\nu_2(k!S(2^{n+1}, k)) = \nu_2(k) + n - (b_{d_2(k)} + \nu_2(k)) + 1 + (k - 2)
= n + k - 1 - b_{d_2(k)} = k - 1. \quad (17)
$$

For $k = 2^{n+1}$, since the factor $i(k - i)$ decreases the 2-adic order the most if we set both $i$ and $k - i$ to $2^{b_d(k)-1} = 2^n$, we get

$$
\nu_2(k!S(2^{n+1}, k)) = \nu_2(k) + n - (b_{d_2(k)} - 1 + \nu_2(k) - 1) + (k - 2)
= n + k - b_{d_2(k)} = k - 1.
$$

**Case 2.** Now we assume that $k \leq 2^n$ and have two subcases. First we discuss the case with $k < 2^n$ provided that $k$ is not a power of two then we consider the case in which $k = 2^m$, $m \leq n$.

Since now $k \leq 2^n$, we need the correction term in (13) which leads to the revised version of (15)

$$
k(k - 1)!S(2^{n+1}, k) = k\frac{(2^{n+1} - 1)!}{(2^n - 1)!^2} \sum_{i=1}^{k-1} \frac{1}{i(k - i)} i!S(2^n, i) (k - i)!S(2^n, k - i)
+ k(k - 1)! \frac{(2^{n+1} - 1)!}{(2^n - 1)!^2} \sum_{j=k}^{2^n} 2^{(j - 1)} B_{2n+1-j} S(j, k) \quad (18)
$$

by setting $k$ and $m$ to $2^n$ and switching from $r$ to $k$ in (13). We proceed similarly to (16) and (17), but this time the correction term in (18) will determine the exact 2-adic order. Clearly, the factor $\binom{2^n - 1}{j-1}$ in the correction term is odd for any $j, k \leq j \leq 2^n$, by Theorem 3.
If \( k < 2^n \) then \( b_{d_2(k)} \leq n - 1 \). If \( k \) is not a power of two then the right hand sides of (16) and (17) become \( n + k - 1 - b_{d_2(k)} \geq k \). Therefore, the first term on the right hand side of (18) contributes an integer multiple of \( 2^k \) to (18). On the other hand, the correction term of (18) will guarantee that \( \nu_2(k!S(2^{n+1}, k)) \) stays at \( k - 1 \). Indeed, the 2-adic order of the \( j \)-th term of the correcting sum is at least \( (k - d_2(k)) + n + (1 + \nu_2(B_{2^n+1-j}) - \nu_2(j)) + (d_2(k) - d_2(j)) \geq n + (k - 1) + (1 - \nu_2(j) - d_2(j)) = n + (k - 1) - d_2(j - 1) \) by Theorem 4 and the fact that \( \nu_2(B_n) \geq -1 \). For the smallest possible value we have that

\[
\min_{k \leq j \leq 2^n} n + (k - 1) - d_2(j - 1) = k - 1 \quad (19)
\]
taken uniquely at \( j = 2^n \). In this case the two inequalities above become equalities since \( \nu_2(S(2^n, k)) = d_2(k) - 1 \) and \( \nu_2(B_{2^n}) = -1 \). Thus, \( \nu_2(k!S(2^{n+1}, k)) = k - 1 \).

We are left with the subcases in which \( k \) is a power of two. The statement is trivially true for \( k = 1 \). If \( k = 2^m \) with \( 1 \leq m \leq n \) then \( b_{d_2(k)} = \nu_2(k) = m \) and the right hand side of (17) changes to

\[
\nu_2(k) + n - (b_{d_2(k)} - 1 + \nu_2(k) - 1) + (k - 2)
\]

\[
= n - m + k \geq k
\]

with \( \max_{1 \leq i \leq k-1} \nu_2(i(k-i)) = b_{d_2(k)} - 1 + \nu_2(k) - 1 \) and the unique optimum is taken at \( i = k - i = 2^{m-1} \). For the correction term, (19) applies again with the same reasoning as above.

We can generalize the above proof to obtain an alternative proof of Theorem 2 although it requires a modified version of inequality (5) of Theorem 4 cf. [7, Remark 2 and Theorem 6] in a somewhat relaxed form:

**Theorem 12.** For \( c \geq 3 \) odd, we have

\[
\nu_2(S(c2^n, k)) \geq d_2(k) - 1, \ 1 \leq k \leq 2^{n+1}. \quad (20)
\]

Below, for any integer \( a \geq 1 \), we use the following simple fact that

\[
d_2(a - 1) = d_2(a) - 1 + \nu_2(a). \quad (21)
\]

This implies \( d_2(c2^n - 1) = d_2(c - 1) + n \) and thus,

\[
d_2(c2^{n+1} - 1) = d_2(c2^n - 1) + 1 = d_2(c) + \nu_2(c) + n. \quad (22)
\]
Proof of Theorem 2. We may assume that $c$ is an odd integer, otherwise we can factor $c$ into a power of two and an odd integer, and $k$ still satisfies $1 \leq k \leq 2^n$. We use induction on $c$ and $n$. Assume that $\nu_2(k!S(s2^t, k)) = k - 1, 1 \leq k \leq 2^t$, for all $1 \leq s \leq c$ and $0 \leq t \leq n$, and prove that it also holds for $t = n + 1$. Then we prove that it also holds for the odd number $s = c + 2$.

The base case with $c = 1$ is covered by the above proof of Theorem 1. Let us assume that $c \geq 3$. Clearly, $d_2(c) \geq 2$. The case with $n = 0$ is trivial since $\nu_2(S(c, 1)) = 0$. Similarly to (18), we get

\[
k(k - 1)!S(c2^{n+1}, k) = k \frac{(c2^{n+1} - 1)!}{(c2^n - 1)!^2} \sum_{i=1}^{k-1} \frac{1}{i(k - i)} i!S(c2^n, i) (k - i)!S(c2^n, k - i)
\]

\[+ k(k - 1)\frac{(c2^{n+1} - 1)!}{(c2^n - 1)!^2} \sum_{j=k}^{c2^n} 2^{c2^n - 1} \frac{(c2^n - 1)}{j} S(j, k)
\]

(23)

by setting $k = m = c2^n$ and switching from $r$ to $k$ in (13). We will see that the correction term in (23) determines the exact 2-adic order. In fact, the first term’s 2-adic order is at least

\[
\nu_2(k) + (\log_2 k) + \nu_2(k - 1), \quad \text{if } k \geq 2 \text{ is odd or even but not a power of two}
\]

\[2\nu_2(k) - 2, \quad \text{if } k \geq 2 \text{ is a power of two},
\]

by (22) and Theorem 12, thus it is at least $k$. Note that the first term disappears if $k = 1$, and the statement $\nu_2(S(c2^{n+1}, 1)) = 0$ is trivial.

If $j$ is odd then the corresponding Bernoulli number $B_{c2^{n+1}-j}$ in the correction term (23) is 0. If $j$ is even then we define $A$ as the 2-adic order of the $j$th term, and we have that

\[
A = \nu_2(k!) + \nu_2((c2^{n+1} - 1)!)) - 2\nu_2((c2^n - 1)!
\]

\[+ (1 + d_2(j - 1) + d_2(c2^n - j) - d_2(c2^n - 1) - 1 - \nu_2(c2^{n+1} - j)) + \nu_2(S(j, k))
\]

\[= (k - d_2(k)) + c2^{n+1} - 1 - d_2(c2^{n+1} - 1) - 2(c2^n - 1 - d_2(c2^n - 1))
\]

\[+ (d_2(j - 1) + d_2(c2^n - j) - d_2(c2^n - 1) - \nu_2(c2^{n+1} - j)) + \nu_2(S(j, k))
\]

\[= k + d_2(j - 1) + d_2(c2^n - j) - \nu_2(c2^{n+1} - j) + \nu_2(S(j, k)) - d_2(k)
\]

\[= k - 1 + \nu_2(j) + 2^{c2^n - 1} - \nu_2(c2^{n+1} - j) - (\nu_2(S(j, k)) - d_2(k) + d_2(j))
\]

by $\nu_2(B_{c2^{n+1}-j}) = -1$, (21), and (22).

Now we prove that the last quantity is at least $k - 1$, and the unique value of $j$ that achieves this lower bound is $j = c \mod 2^{\log_2 c}$, i.e., when we remove the most
significant binary digit of $c$. We set $j = c'2^{n+q}$ with $c'$ odd and $k \leq j \leq c2^n$ and identify four cases according to the value of $q$.

If $-n \leq q < 0$ then
\[
A \geq k - 1 + n + q + d_2(c2^{-q} - c') - (n + q) \geq k
\]
bys by (5) and since $c' \neq c2^{-q}$, i.e., $j \neq c2^n$.

If $q = 0$, i.e., $j = c'2^n$, then
\[
A \geq k - 1 + n + d_2(c - c') - n + (d_2(k) - 1 - d_2(k) + d_2(c'))
\]
\[
\geq k - 1 + d_2(c) - 1 \geq k
\]
by Theorem 12.

If $q = 1$ then $2c' < c$ and
\[
A = k - 1 + n + 1 + d_2(c - 2c') - n_2(c - c') - (n + 1) + (-1 + d_2(c'))
\]
\[
= k - 1 + d_2(c) - 1 + n_2(c - c') \geq k - 1
\]
by the induction hypothesis as $c' < c$ and $1 \leq k \leq 2^{n+1}$ imply that $n_2(S(c'2^{n+1}, k)) = d_2(k) - 1$. It is easy to prove, e.g., by induction on the number of blocks of zeros in the binary representation of $c$, that $A$ can reach the lower bound $k - 1$ exactly if $c'$ is derived from $c$ by removing its most significant binary digit. By the way, if $c'' = c2^{\lceil \log_2 c \rceil - i}$ with $0 \leq i \leq \lfloor \log_2 c \rfloor - 1$, then $d_2(c) - 1 + n_2((c''_2) - n_2(c - c'')$ is equal to the number of ones in $c2^{\lceil \log_2 c \rceil - c''}$.

If $q \geq 2$ then by (5) we get that
\[
A \geq k - 1 + n + q + d_2(c - c'2^q) - (n + 1) \geq k - 1 + q - 1 \geq k.
\]
The proof of $n_2(k!S(c2^{n+1}, k)) = k - 1$ for $1 \leq k \leq 2^{n+1}$ and $n \geq 0$ is complete for $c$, and now we can proceed with the next odd $c$.

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References


