Constraints on an empirical equation for asymmetry-induced transport

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Previous work on asymmetry-induced transport in a modified Malmberg–Penning trap showed that the radial particle flux was empirically constrained to be of the form $\Gamma(e) = -(B_0/B)^{1.33} D(e) \nabla n_0 + f(e)$, where $\epsilon = \omega - l \omega_R$ is the column rotation frequency, $\omega$ and $l$ are the asymmetry frequency and azimuthal mode number, $\nabla n_0$ is the radial density gradient, $B$ is the magnetic field, $B_0$ is an empirical constant, and $D(e)$ and $f(e)$ are unknown functions. In this paper, it is shown that further constraints can be placed upon $D(e)$ and $f(e)$ by comparing data near the $\epsilon = 0$ points to a first order expansion of $\Gamma(e)$. It is shown that $dD/de(0) \neq 0$, in contradiction to resonant particle theory, and that $f(e)$ can only be a fraction of the size predicted by that theory. Finally, it is shown that $dD/de(0)$ exhibits a power-law scaling with radius, magnetic field, and the bias of the center conductor of the trap. © 2010 American Institute of Physics. [doi:10.1063/1.3381069]

I. INTRODUCTION

Asymmetry-induced radial particle transport in cylindrical Malmberg–Penning non-neutral plasma traps has been studied for over 25 years by several research groups.\textsuperscript{1–8} Despite the simplicity of these traps, a full understanding of the transport remains elusive, and there is little agreement between experiments and theory. Indeed, our previous work in a modified version\textsuperscript{9} of these traps designed specifically to test resonant particle transport theory\textsuperscript{10} revealed serious discrepancies between experiments and this theory.\textsuperscript{11} In particular, the experimental dependence of the transport on asymmetry frequency $\omega$ was both quantitatively and qualitatively different than the predictions of theory. However, it was not clear from these experiments which part(s) of the theory were in error.

In a previous paper,\textsuperscript{12} we presented a new approach to the study of asymmetry-induced transport based on the hypothesis that the asymmetry frequency $\omega$ and the plasma rotation frequency $\omega_R$ always enter the physics in the combination $\omega - l \omega_R$, where $l$ is the azimuthal mode number of the asymmetry. This hypothesis was suggested by the appearance of this combination in theories for various phenomena in non-neutral plasmas. In order to simplify the data analysis, it proved useful to focus on data points where this combination was zero. From our typical measurements of the radial particle flux $\Gamma$ versus radius $r$, we selected the flux $\Gamma_{sel}$ at the radius where $\omega - l \omega_R = 0$. By independently varying $\omega$, $\omega_R$, and the magnetic field $B$, we showed that $\Gamma_{sel}$ for a fixed asymmetry amplitude, satisfied the equation

$$\Gamma_{sel} = -(B_0/B)^{1.33} D_0 \nabla n_0 + f_0,$$

where $\nabla n_0$ is the radial density gradient, and $B_0 = 233$ G, $D_0 = 1.00$ cm$^2$ s$^{-1}$, and $f_0 = -1.01 \times 10^5$ cm$^{-4}$ are empirical constants. Although this equation only gave the flux for points where $\omega - l \omega_R = 0$, we also deduced the form of the general flux equation. Our data analysis revealed that the expression for the general flux $\Gamma$ must be a function of the combination $\omega - l \omega_R$. In addition, the general data flux expression had to reduce to the equation for $\Gamma_{sel}$ when $\omega - l \omega_R = 0$. Without further information, we thus allowed both $D_0$ and $f_0$ to become functions of $\epsilon = \omega - l \omega_R$, obtaining

$$\Gamma(e) = -(B_0/B)^{1.33} D(e) \nabla n_0 + f(e),$$

where $D(e)$ and $f(e)$ are unknown functions and $D(e=0) = D_0$ and $f(e=0) = f_0$.

While Eq. (2) appears as a slight modification of Eq. (1), the consideration of nonzero $\epsilon$ opens the possibility for additional parametric dependences that do not appear in Eq. (1) if these parameters appear in a product or quotient with $\epsilon$. For example, if $\epsilon$ is scaled to the cyclotron frequency $\omega_c$ (i.e., if it appears as $\epsilon / \omega_c$), this would introduce an additional dependence on $B$ not captured in the $B^{-1.33}$ scaling of Eq. (1). Similarly, if $\epsilon$ appears scaled to $\omega_R$, then our initial hypothesis about the rotation frequency only appearing in the combination $\omega - l \omega_R$ will have to be modified.

The magnetic field dependence of Eq. (2) does not match that given by resonant particle transport theory in either the plateau or banana regimes. The rest of the equation, however, is compatible with that theory. In particular, the theory gives, for either regime,

$$f(e) = n_0 \frac{dT}{dr} \left( \frac{e^2}{2 k^2 v^2} - \frac{1}{2} \right) + \frac{r n_0 \omega_c \epsilon}{lv^2},$$

while $D(e) = \bar{D} \exp(-\epsilon^2 / 2 k^2 v^2)$, where $k$ is the axial wavenumber of the asymmetry, $v$ is the thermal velocity, $\omega_c$ is the cyclotron frequency, and $\bar{D}$ depends on the regime.

In this paper, we show that further empirical constraints can be placed on $D(e)$ and $f(e)$ by examining data points adjacent to the $\epsilon = 0$ points and comparing them to a first order expansion of $\Gamma(e)$. In particular, we show that $dD/de(0) \neq 0$, which excludes $D(e)$ of the form given by resonant particle theory. Further, we show that Eq. (3) is also incompatible with the data unless it is reduced in magnitude. Finally, we show that $dD/de(0)$ has power-law dependences on $r$, $B$, and the center wire bias $\phi_{cw}$.  

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II. LINEARIZATION

We start by performing an expansion of Eq. (2) for small values of \( \epsilon \). The expansions for \( D(\epsilon) \) and \( f(\epsilon) \) are

\[
D(\epsilon) = D(0) + \epsilon \frac{dD}{d\epsilon}(0) + \frac{\epsilon^2}{2} \frac{d^2D}{d\epsilon^2}(0) + \cdots
\]

and

\[
f(\epsilon) = f(0) + \epsilon \frac{df}{d\epsilon}(0) + \frac{\epsilon^2}{2} \frac{d^2f}{d\epsilon^2}(0) + \cdots.
\]

Using these in Eq. (2), we obtain, to first order in \( \epsilon \),

\[
\Gamma = -\left( \frac{B_0}{B} \right)^3 \left\{ D_0 [\nabla n_0 + f_0] + \epsilon \left( \nabla n_0 + f_0 \right) \right\},
\]

where we have used the equivalent notations \( D(\epsilon=0) = D_0 \) and \( f(\epsilon=0) = f_0 \). Any variation in parameters \( \Omega \) multiplying \( \epsilon \) will be higher order in \( \epsilon \) since \( d(\epsilon \Omega) = \Omega d\epsilon + \epsilon d\Omega \). Noting that the zeroth-order term in the expansion is \( \Gamma_{\text{sel}} \), as given by Eq. (1), we move this term to the left hand side. Taking the derivative with respect to \( \epsilon \) we obtain

\[
\frac{d(\Gamma - \Gamma_{\text{sel}})}{d\epsilon} = -\left( \frac{B_0}{B} \right)^3 \left[ (\nabla n_0 + f_0) \frac{dD}{d\epsilon}(0) + D_0 \frac{df}{d\epsilon}(0) \right].
\]

From this result, we can see that since \( \nabla n_0, D_0, \) and \( f_0 \) are known, we may be able to obtain information about \( (dD/d\epsilon)(0) \) and \( (df/d\epsilon)(0) \) by examining the experimental values of \([d(\Gamma - \Gamma_{\text{sel}})]/d\epsilon\) for small \( \epsilon \).

III. EXPERIMENTAL APPARATUS

Our transport studies are performed in the modified Malmberg–Penning trap shown in shown in Fig. 1. As in the standard trap design, a uniform axial magnetic field \( B \) provides radial confinement of injected electrons, while negatively biased end cylinders (the injection gate and dump gate) provide axial confinement. Our device also operates in the standard inject-hold-dump cycle. A cycle begins by grounding the injection gate which allows electrons from the gun to flow into the central region. This injection gate is then returned to a negative bias which traps the electrons. After a chosen period of time, the dump gate is grounded and the electrons leave the trap and hit a positively biased phosphor screen. Analysis of the images on this screen provides the primary diagnostic.

The principal modification in our device is replacing the usual plasma column with a biased wire running along the axis of the trap. The wire provides a radial electric field to replace the field normally produced by the plasma column and allows the injected low density electrons to have the same zeroth-order dynamical motions (axial bounce and azimuthal \( E \times B \) drift motions) as in a standard trap. The lower density (10^5 cm^-3) and high temperature (4 eV) of the electrons give a Debye length larger than the trap radius. Under these conditions, potentials in the plasma are essentially the vacuum potentials and previously encountered\(^5\) complications due to collective effects are minimized.\(^{11}\) Our design also allows the drift rotation frequency \( \omega_B(r) \) to be easily adjusted by varying the center wire bias \( \phi_{\text{cw}} \) since

\[
\omega_B = \frac{-\phi_{\text{cw}}}{r^2 B \ln(R/a)},
\]

where \( R \) and \( a \) are the radii of the wall and the center wire, respectively. Despite these changes, the unperturbed confinement time has similar magnitude and shows\(^9\) the same \((L/B)^2\) scaling found in higher density experiments, thus supporting the idea that the radial transport is primarily a single particle process and confirming the relevance of our experiments to standard trap physics.

A unique feature of our device is that the entire confinement region is sectored (five cylinders, labeled S1 through S5 in Fig. 1, with eight azimuthal divisions each). This allows us to apply a simple, known asymmetry by selecting the amplitude and phase of the voltages applied to each sector to produce a helical standing wave of the form

\[
\phi(r, \theta, z, t) = \phi_W \left( \frac{r}{R} \right)^l \cos \left( \frac{n \pi z}{L} \right) \cos(\theta - \omega t),
\]

where \( \phi_W \) is the asymmetry potential at the wall (typically 0.2 V), \( R \) is the wall radius (3.82 cm), \( L \) is the length of the confinement region (76.8 cm), and \( n \) and \( l \) are the axial and azimuthal Fourier mode numbers, respectively, and here \( z \) is measured from one end of the confinement region. For these experiments \( n=l=1 \) and the relative phases of the applied voltages are adjusted so that the asymmetry rotates in the same direction as the zeroth-order azimuthal \( E \times B \) drift. For these experiments, the higher order harmonics of the applied asymmetry have amplitudes less than 10% of the fundamental. Since the transport typically scales like the square of the asymmetry amplitude,\(^{13}\) the effect of these harmonics can be ignored.

Data acquisition for these transport studies can be summarized as follows; details have been given elsewhere.\(^{3,11}\) Electrons injected into the trap from an off-axis gun are quickly dispersed into an annular distribution. At a chosen time (here, 1600 ms after injection), the asymmetries are switched on for a period of time \( \delta t \) (here, 100 ms) and then switched off. At the end of the experiment cycle, the electrons are dumped axially onto a phosphor screen and the resulting image is digitized using a 512×512 pixel charge-
coupled device camera. A radial cut through this image gives the density profile \( n_0(r) \) of the electrons. A typical profile is shown in Fig. 2. Shot-to-shot variation in the number of injected electrons is less than 1% and the data is very reproducible. Calibration is provided by a measurement of the total charge being dumped. Profiles are taken both with the asymmetry on and off, and the resulting change in density \( \Delta n_0(r) \) is obtained. The background transport is typically small compared to the induced transport and is subtracted off. If the asymmetry amplitude is small enough and the asymmetry pulse length is short enough, then \( \Delta n_0(r) \) will increase linearly in time. We may then approximate the asymmetry frequency matches the rotation frequency, i.e., \( f = \omega \) for three representative asymmetry frequencies. Each of the 26 frequencies \( f \) are used to construct the radial flux \( \Gamma(r) \) for the same three representative frequencies. The radius values in each plot are mapped to the corresponding values of \( r/\omega \) to produce this plot. The slope of these curves at the origin is used to obtain the left hand side of Eq. (7). A peculiarity result. Now the \( \Gamma - \Gamma_{sel} \) curves shown in Fig. 3(b) can be constructed by subtracting \( \Gamma_{sel}(r) \) from each of the \( \Gamma(r) \) curves. Finally, we use Eq. (8) to map the radii in Fig. 3(b) to the corresponding values of \( \omega - \omega_R \), again doing this

FIG. 2. A typical density profile taken 1600 ms after injection.
separately for each asymmetry frequency. For our representative cases this yields the \( \Gamma - \Gamma_{\text{rel}} \) versus \( \omega - \log \) curves shown in Fig. 3(c). We then find the slope of these curves at the point where they pass through the origin to obtain \( d(\Gamma - \Gamma_{\text{rel}})/d\epsilon \) for small \( \epsilon \). These values are then plotted versus the radius at which \( \epsilon = 0 \). Repeating this process for all the asymmetry frequencies produces the black dots in Fig. 4.

To estimate the uncertainty in \( d(\Gamma - \Gamma_{\text{rel}})/d\epsilon \), we have obtained this quantity in two ways. First, we simply take one data point on either side of \( \epsilon = 0 \) and approximate \( d(\Gamma - \Gamma_{\text{rel}})/d\epsilon \cong \Delta (\Gamma - \Gamma_{\text{rel}})/\Delta \epsilon \). Second, we take three data points on either side of \( \epsilon = 0 \), fit a second order polynomial to the data, and use the linear coefficient for \( d(\Gamma - \Gamma_{\text{rel}})/d\epsilon \). We used the result of this latter procedure for the data points and the difference between the two procedures for the uncertainty.

Given the form of Eq. (7), it is instructive to compare the data points in Fig. 4 with the data for \(- (\nabla n_0 + f_0)\), and this latter quantity is shown by the solid line. Note that both the black dots and the solid line have a zero-crossing at essentially the same radius. Since most quantities that enter the physics do not change sign with radius, this suggests that the first term in Eq. (7) is dominant, at least in the vicinity of the zero-crossing radius. This conclusion is supported when the experimental parameters are varied. Varying the magnetic field \( B \) and the center wire bias \( \phi_{\text{cw}} \), we obtain results similar to Fig. 4 and shifts in the zero-crossing of \(- (\nabla n_0 + f_0) \) (due to the variation in parameters) are tracked by the zero-crossing of \( d(\Gamma - \Gamma_{\text{rel}})/d\epsilon \). This correlation is shown in Fig. 5 where the ordinate gives the scaled radius \( r/R \) where \( d(\Gamma - \Gamma_{\text{rel}})/d\epsilon \) changes sign and the abscissa gives the scaled radius where \(- (\nabla n_0 + f_0) \) changes sign. The uncertainties stem from the uncertainties in the data plots such as that of Fig. 4. Data are given for three values of the center wire bias \( \phi_{\text{cw}} \) for each of four magnetic fields \( B \), with the higher absolute values of \( \phi_{\text{cw}} \) giving zero-crossings at larger \( r/R \) as indicated by the arrow.

The observed correlation allows us to restrict the form of the unknown function \( D(\epsilon) \). Referring to Eq. (7), we see that the correlation requires that \( dD/d\epsilon(0) \neq 0 \). This is noteworthy since it excludes \( D(\epsilon) \) of the form found in resonant particle transport theory.\(^\text{11}\) In this theory, \( D(\epsilon) \propto \exp(-\epsilon^2/2\theta^2) \) and thus \( dD/d\epsilon(0) = 0 \). Here \( k = n\pi/L \) is the axial mode number.

The correlation between zero-crossings also shows that the form of \( f(\epsilon) \) given by resonant particle theory is incorrect. According to the theory, for either the plateau or banana regimes, \( f(\epsilon) \) is given by Eq. (3) and thus \( \frac{df}{d\epsilon}(0) = \frac{r n_0 \omega}{\theta^2} \).

The effect of including such a term is shown in Fig. 6. Using experimental values in Eq. (11) we obtain the curve for \( D_0(df/d\epsilon)(0) \) shown in Fig. 6(a). Since we do not know \( (df/d\epsilon)(0) \), we move \( D_0(df/d\epsilon)(0) \) to the left hand side of Eq. (7):

\[
\left(\frac{B}{B_0}\right)^{1.33} \frac{d(\Gamma - \Gamma_{\text{rel}})}{d\epsilon} + D_0 \frac{df}{d\epsilon}(0) = -(\nabla n_0 + f_0) \frac{dD}{d\epsilon}(0). 
\]

(12)

Figure 6(b) shows the effect of including the \( D_0(df/d\epsilon)(0) \) term. The lower line shows the same data plotted as dots in Fig. 4 multiplied by the constant \( (B/B_0)\). We have interpolated the data between the points to allow for easy addition of the \( D_0(df/d\epsilon)(0) \) data which includes 256 radial values. The upper line shows the effect of this addition. In this case, the curve is shifted upward enough so that there is no longer a zero-crossing. Since we know that the right side of Eq. (12) crosses zero at roughly \( r/R = 0.4 \), we have a contradiction. Thus, \( D_0(df/d\epsilon)(0) \) cannot be given by Eq. (11).

It is tempting at this point to conclude that \( D_0(df/d\epsilon)(0) \) is zero, but there are several reasons to resist this conclusion. The data from previous experiments\(^\text{11}\) on the frequency dependence of the flux at the point where \( \nabla n_0 = 0 \) require that there be a second term in the flux equation that is a function of frequency. Also, the theoretical origin\(^\text{10}\) of this term seems fairly robust and reflects a mobility contribution to the transport. Finally, the efficacy of the rotating-
fraction of $D_0(df/de)(0)$ to the original data, the upper curve in Fig. 6(b) will be shifted up by a lesser amount, with the result that the zero-crossing point will be shifted to the left to lower values of $r/R$. Referring to Fig. 5, we see that such an adjustment will move the data points down and improve the correlation for some of the data. Of course, if $\eta$ is too large that the shift will make the correlation worse. We find that $\eta=0.15$ produces the best result with the standard deviation from a perfect correlation improving from $\sigma_{r/R}=0.024$ (for $\eta=0$) to 0.017. Values of $\eta$ greater than 0.31 degrade the correlation, producing $\sigma_{r/R}$ values greater than 0.024.

So far we have drawn our conclusions solely from the correlation in the zero-crossings of the two sides of Eq. (12). We now examine the full set of $d(\Gamma-\Gamma_{sel})/de$ versus $r/R$ data and find parameter scalings in $(df/de)(0)$. When data was taken for various values of the center wire bias $\phi_{cw}$ and magnetic field $B$, the data points in Fig. 4 increased in magnitude with $B$ and decreased with $\phi_{cw}$. It also is apparent from this figure that the solid line data will have to be multiplied by an increasing function of radius to match the dot data. To quantify these scalings, we have assumed a power law dependence on $r$, $B$, and $\phi_{cw}$, i.e.,

$$\frac{dD}{de}(0) = C \left(\frac{r}{R}\right)^a \left(\frac{B}{B_0}\right)^b \left(\frac{\phi_{cw}}{\phi_0}\right)^c,$$

and adjusted the exponents $a$, $b$, and $c$ and the multiplicative factor $\eta$ through multiple regression to achieve the best correlation between the two sides of the equation

$$\left(\frac{B}{B_0}\right)^{1.33} \frac{d(\Gamma-\Gamma_{sel})}{de}(0) + \eta D_0 \frac{df}{de}(0) = -(\nabla n_0 + f_0) \frac{dD}{de}(0).$$

In Eq. (13), $C$ is a constant to be determined and $\phi_0=-140$ V. For the regression, we use the correlation coefficient $r_{xy}$ as the figure of merit with $r_{xy}=1$ corresponding to perfect correlation. The results of the regression for various cases are shown in Table I. The first case ("all free") gives the result when all four parameters ($\eta$, $a$, $b$, and $c$) were adjusted to give the largest value of $r_{xy}$. The next two columns give the result of fixing the value of $\eta$ and adjusting $a$, $b$, and $c$. The final four columns give the results when all four parameters are fixed. In addition to $r_{xy}$, the table also

![FIG. 6. (a) $D_0(df/de)(0)$ as given by resonant particle theory [Eq. (11)] vs scaled radius $r/R$. (b) The effect of adding $D_0(df/de)(0)$ to $(B/B_0)^{1.33} d(\Gamma-\Gamma_{sel})/de$. The data has been interpolated to match to 256 radial positions of the $D_0(df/de)(0)$ data. The upper line shows the same data with $D_0(df/de)(0)$ added. In this case the former zero-crossing point at $r/R = 0.4$ no longer occurs.](image)

<table>
<thead>
<tr>
<th>Case $→$</th>
<th>All free</th>
<th>Fixed $\eta$</th>
<th>Fixed $\eta$</th>
<th>All fixed</th>
<th>All fixed</th>
<th>All fixed</th>
<th>All fixed</th>
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<tbody>
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<tr>
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<td>2.77</td>
<td>3.04</td>
<td>2.95</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$b$</td>
<td>1.54</td>
<td>1.62</td>
<td>1.59</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>$c$</td>
<td>0.95</td>
<td>1.03</td>
<td>1.01</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>1</td>
</tr>
<tr>
<td>$r_{xy}$</td>
<td>0.929</td>
<td>0.916</td>
<td>0.923</td>
<td>0.877</td>
<td>0.900</td>
<td>0.902</td>
<td>0.922</td>
</tr>
<tr>
<td>$\Delta r_{xy}$</td>
<td>...</td>
<td>0.013</td>
<td>0.006</td>
<td>0.052</td>
<td>0.029</td>
<td>0.027</td>
<td>0.007</td>
</tr>
<tr>
<td>$C (10^{-5} \text{ cm}^2/\text{rad})$</td>
<td>0.384</td>
<td>0.365</td>
<td>0.371</td>
<td>0.363</td>
<td>0.585</td>
<td>0.294</td>
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<tr>
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<td>1.75</td>
<td>1.67</td>
<td>2.17</td>
<td>1.94</td>
<td>1.91</td>
<td>1.61</td>
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</table>
These values are close to those that would be obtained if the correlation discussed previously. The remaining columns of Table I, for example, allow such functions if they are centered at another point. Indeed, our simulation of the particle dynamics in asymmetry-induced transport suggests that this may be so. We do not know of an analytic technique that would allow such functions that would fit the data, but without success.

VI. CONCLUSION

We have shown that further constraints can be placed upon \( D(e) \) and \( f(e) \) by comparing data near the \( e=0 \) points to a first order expansion of \( \Gamma(e) \). The analysis revealed that \( dD/de(0) \neq 0 \), in contradiction to resonant particle theory, and that \( f(e) \) can only be a fraction of the size predicted by that theory. We have also shown that \( dD/de(0) \) exhibits a power-law scaling with radius, magnetic field, and the bias of the center conductor of the trap. These results provide a touchstone for future theoretical developments.

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