8.12  

a) Eqn. 8.32 gives

\[ U_{eff}(r) = -\frac{G m_1 m_2}{r} + \frac{l^2}{2 r^2} \]

There is a fixed radius orbit where \( dU_{eff}/dr = 0 \)

\[ \frac{dU_{eff}}{dr} = \frac{G m_1 m_2}{r^2} - \frac{l^2}{mr^3} = 0 \]

Solving for \( r \), we have \( r = \frac{l^2}{G m_1 m_2 \mu} = r_0 \)

b) For stability, check \( \frac{d^2 U_{eff}}{dr^2}(r_0) \)

\[ \frac{d^2 U_{eff}}{dr^2} = \frac{2 G m_1 m_2}{r^3} + \frac{3l^2}{mr^4} = \frac{1}{r^3} \left[ 2 G m_1 m_2 + \frac{3l^2}{\mu} \right] \]

\[ \uparrow \frac{1}{r_0^3} \left[ -2 G m_1 m_2 + \frac{3l^2}{\mu} - G m_2 m_2 \mu \right] = \frac{1}{r_0^3} G m_1 m_2 \]

Since this last result is always positive, the orbit is stable.

To find the period of oscillations around this equilibrium, note that from Eqn. 5.29

\[ \mu \ddot{r} = -\frac{d}{dr} U_{eff}(r) \]

\[ = -\frac{1}{r^3} \left[ G m_1 m_2 r - \frac{l^2}{\mu} \right] \]

Write \( r = r_0 + \epsilon \) where \( \epsilon \) is small. Since \( r_0 \) is constant;

\[ \mu \dot{\epsilon} \propto \frac{1}{r_0^3} \left[ G m_1 m_2 (r_0 + \epsilon) - \frac{l^2}{\mu} \right] = \frac{1}{r_0^3} G m_1 m_2 \epsilon \]

This is the SDO equation. Thus

\[ \omega^2 = \frac{G m_1 m_2}{\mu r_0^3} \]

This is the angular frequency of the orbit. For a circular

\[ \text{orbit} \quad F_{grav} = m a = \mu \frac{v^2}{r} = \mu \omega^2 r \]

\[ \omega^2 = \frac{F_{grav}}{\mu r} = \frac{G m_1 m_2}{\mu r^2} \]

\[ F_{grav} = \frac{G m_1 m_2}{r^2} \]
Equ. 8.50 gives $r_{\text{min}} = \frac{c}{1+\varepsilon}$ and $r_{\text{min}}$ is the perigee. So $E = \frac{c}{r_{\text{min}}} - 1$. Equ 8.48 gives $C = \frac{1}{2} \frac{m}{\mu}$, $l = m \sqrt{r_{\text{min}}}$ and $V = \frac{GM_m}{r_{\text{min}}}$. \\
At perigee
\[ C = \frac{m^2 \frac{V^2}{l^2}}{GM_m m/(m+M_e)} = \frac{V^2}{GM_e} \frac{r_{\text{min}}^2}{(m+M_e)} \approx \frac{V^2}{GM_e} \]
\[ \text{since } m \gg M_e \]

Noting $r_{\text{min}} = R_e + h_{\min}$, we can re-write this as
\[
C = \frac{V^2 \left(1 + \frac{h_{\min}}{R_e}\right)^2}{GM_e/R_e^2} = \left(3500 \text{ m/s}\right)^2 \left(1 + \frac{250 \times 10^3 \text{ m}^2}{6.4 \times 10^2 \text{ m}^2}\right)
\]
\[ C = 7.96 \times 10^6 \text{ m} \]

and thus $E = \frac{C}{R_e + h_{\min}} - 1 = 0.197$ \checkmark

Then
\[
V_{\text{max}} = \frac{C}{1 - E} = 9.41 \times 10^6 \text{ m}
\]
and $h_{\text{max}} = V_{\text{max}} - R_e = 3.51 \times 10^6 \text{ m}$ \checkmark