7.37 a) \[ u = 0 \]

For \( m_1 \), in polar co-ords, \[ v^2 = r^2 + r^2 \dot{\phi}^2 \]

For \( m_2 \), \[ v = \frac{d}{dt}(L-r) = -\dot{r} \] so \[ v^2 = \dot{r}^2 \]

Thus \[ L = T - U = \frac{1}{2} m_1 (r^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 + mg(L-r) \]

or since \( m_1 = m_2 = m \)

\[ L = mr^2 + \frac{1}{2} mr^2 \dot{\phi}^2 - mg r + mg L \]

b) \[ \frac{\partial L}{\partial \dot{r}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0 \]

\[ mr^2 \dot{\phi} - mg - \frac{d}{dt}(2mr\dot{r}) = 0 \]

\[ \ddot{r} = -\frac{1}{2} g + \frac{1}{2} r \dot{\phi}^2 \]

\[ \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \]

\[ 0 - \frac{d}{dt}(mr^2 \dot{\phi}) = 0 \rightarrow mr^2 \dot{\phi} = \text{const.} \]

Since \( r \dot{\phi} = V \), \( mr^2 \dot{\phi} = r m V = r \rho = |r \times \vec{p}| = |\vec{L}| \)

So this equation of motion expresses conservation of angular momentum.

b) \( mr^2 \dot{\phi} = l \rightarrow \dot{\phi} = \frac{l}{mr^2} \)

Then

\[ \ddot{r} = -\frac{1}{2} g + \frac{1}{2} \frac{L^2}{mr^2} \]

For circular path, \( \dot{r} = 0 \) so \( g = \frac{L^2}{mr^2} \)

or

\[ mg = \frac{L^2}{mr^2} = \frac{(r m V)^2}{mr^2} = m \frac{V^2}{c_0} \]

\( \rightarrow \) tension provided force for centripetal acceleration.
\[ r = r_0 + \epsilon \]
\[ \ddot{r} = \frac{1}{2} g + \frac{1}{2} \frac{L^2}{m^2 r^3} \]
\[ \ddot{\epsilon} = \frac{1}{2} g + \frac{1}{2} \frac{L^2}{m^2 r^3} \left( 1 - 3 \frac{\epsilon}{r_0} \right) \]

but \( mg = \frac{L^2}{m r_0^3} \) so

\[ \ddot{\epsilon} = -\frac{3}{2} \frac{L^2}{m r_0^3} \epsilon \]

\[ \rightarrow \text{This is the SHO equation with } W = \sqrt{\frac{3}{2} \frac{L^2}{m r_0^3}}. \]

We know this is stable.

\[ \begin{align*}
7.41 \quad & \text{In cylindrical coords} \\
& \mathbf{v} = (\dot{\varphi}, \rho \dot{\varphi}, \dot{z}) \quad \text{and} \\
& V^2 = \dot{\varphi}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2 \quad \text{but } \dot{z} = 2k \ddot{\varphi} \text{ so} \\
L = T - U = \frac{1}{2} m (\dot{\varphi}^2 + \rho^2 \dot{\varphi}^2 + 4k^2 \rho^2 \dot{\varphi}^2) - mgk \ddot{\varphi}^2 \quad \text{valid} \\
\frac{\partial L}{\partial \dot{\varphi}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\varphi}} &= 0 \\
\frac{1}{2} m (2\dot{\varphi}^2 + 8k^2 \rho^2 \dot{\varphi}^2) - 2mgk \ddot{\varphi} - \frac{d}{dt} \left( m \ddot{\varphi}^2 + 4k^2 \rho^2 \ddot{\varphi}^2 \right) &= 0 \\
& \left[ \ddot{\varphi} + 4k^2 (\rho^2 + 2k \ddot{\varphi}) \right] \\
* & \ddot{\varphi} (1 + 4k^2 \rho^2) + 4k^2 \rho \ddot{\varphi} = \rho (\omega^2 - 2kg) \quad \text{valid} \\
\text{Equilibrium means } \ddot{\varphi} = 0 \text{ and } \dot{\varphi} = 0, \text{ hence } \rho (\omega^2 - 2kg) = 0 \end{align*} \]
7.41 (cont) \( f(w^2 - 2kg) = 0 \) can be satisfied with \( f = 0 \)

or \( w^2 = 2kg \).

For \( f = 0 \), \( f \) becomes

\[ \dot{f} = (w^2 - 2kg)f \]

For \( w^2 - 2kg < 0 \), this is stable and stable.

For \( w^2 - 2kg > 0 \), solutions for \( f \) grow exponentially

and thus are unstable.

If \( w^2 - 2kg = 0 \), any value of \( f \) is an equilibrium,

\[ \ddot{f} (1 + 4k^2p^2) + 4k^2p^2 \dot{f}^2 = 0 \]

always positive.

Note that the 2nd term will always be positive. Thus

\( \dot{f} \) will always be negative. If we give the bead

some positive velocity away from equilibrium, the negative

\( \dot{f} \) will slow it until \( \dot{f} = 0 \) and it will rest at

the new equilibrium. It will not return to its

start point. If we give the bead some negative

velocity, \( \dot{f} \) will make it more negative. Again,

it will not return to its start point. Thus none

of these cases are stable.
7.50

Constraint eqn. \( f(x,y) = x + y = \text{const.} \)

\[ L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 + m_2 g y \]

\[ \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{x}} \quad \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{y}} \]

0 + \lambda = m_1 \ddot{x} 

\[ m_2 g + \lambda = m_2 \ddot{y} \]

\[ \lambda = m_1 \ddot{x} \quad \rightarrow \quad m_2 g = m_2 \ddot{y} - m_1 \ddot{x} \]

but from constraint eqn. \( \ddot{x} = -\ddot{y} \), so

\[ m_2 g = (m_2 + m_1) \ddot{y} \]

\[ \ddot{y} = \frac{m_2}{m_2 + m_1} g \checkmark \]

\[ \ddot{x} = -\frac{m_1}{m_2 + m_1} g \checkmark \]

\[ \lambda = -\frac{m_1 m_2}{m_1 + m_2} g \checkmark \]

Eq. 7.122:

\[ F_{\text{str}}^{x} = \lambda \frac{\partial f}{\partial x} = \lambda = \frac{m_1 m_2}{m_1 + m_2} g \]

This is the tension force on \( m_1 \).

Similarly \( F_{\text{str}}^{y} = \lambda \frac{\partial f}{\partial y} = \lambda \) (some tension force on \( m_2 \)).

Newton:

\[ m_2 g - T = m_2 \ddot{y} \quad \rightarrow \quad m_2 g = m_2 \ddot{y} - m_1 \ddot{x} \]

\[ -T = m_1 \ddot{x} \]

and \( \ddot{x} = -\ddot{y} \rightarrow m_2 g = (m_2 + m_1) \ddot{y} \)

\[ \ddot{y} = \frac{m_2}{m_2 + m_1} g \checkmark \]

\[ \ddot{x} = -\frac{m_1}{m_2 + m_1} g \checkmark \]

\[ T = -\frac{m_1 m_2}{m_2 + m_1} g \checkmark \]